Signal Recovery Using Gowers' Norms

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Motivation

Let's send two **messages** using two different approaches:

"Hi"
↓
0100100001101001
↓
■100■0000110■001

No way to restore the original signal :(

"Hi"
↓
0100100001101001
↓
DFT:

[$\widehat{f}_0, \widehat{f}_1, \widehat{f}_2, \widehat{f}_3, \widehat{f}_4, \widehat{f}_5, \widehat{f}_6, \widehat{f}_7, \widehat{f}_8, \widehat{f}_9, \widehat{f}_{10}, \widehat{f}_{11}, \widehat{f}_{12}, \widehat{f}_{13}, \widehat{f}_{14}, \widehat{f}_{15}$]

↓

*R*eceived:

[■, $\widehat{f}_1, \widehat{f}_2, \widehat{f}_3$, ■, ■, $\widehat{f}_6, \widehat{f}_7, \widehat{f}_8, \widehat{f}_9, \widehat{f}_{10}, \widehat{f}_{11}, \widehat{f}_{12}$, ■, $\widehat{f}_{14}, \widehat{f}_{15}$]

We receive only part of a signal/frequencies - the rest is missing.

Questions we want to answer:

- Is it possible to reconstruct the full message?
- What are sufficient conditions for reconstruction?
- What if we know the signal/frequency is 'structured'?

Background

- We will call an arbitrary function $f : \mathbb{Z}_N^d \to \mathbb{C}$ a **signal**.
- We will call an arbitrary function's Fourier transform $\widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ a **frequency**.

Definition: For a function $f : \mathbb{Z}_N^d \to \mathbb{C}$, the **normalized DFT** is:

$$\widehat{f}(k) := N^{-d/2} \sum_{n \in \mathbb{Z}_N^d} f(n)\chi(-kn),$$

where $\chi(x) = e^{-2\pi i k \cdot x/N}$. Then, the inverse transform formula follows:

$$f(n) = N^{-d/2} \sum_{k \in \mathbb{Z}_N^d} \widehat{f}(k)\chi(kn).$$

Classical Uncertainty Principle

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a nonzero function with support $\operatorname{supp}(f) \subseteq \mathbb{Z}_N^d$. Let $\widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ denote the discrete Fourier transform of f, with support $\operatorname{supp}(\widehat{f}) \subseteq \mathbb{Z}_N^d$. Then the following inequality holds:

$$|\operatorname{supp}(f)| \cdot |\operatorname{supp}(\widehat{f})| \geq N^d.$$

Improved uncertainty principle and unique recovery condition

We are interested in working with the sets that have low **additive energy** defined by:

$$\Lambda_2(A) := \left|\left\{(x_1, x_2, x_3, x_4) \in A^4 : x_1 + x_2 = x_3 + x_4\right\}\right|.$$

Using this, J. Iosevich, A. Mayeli et al. in 2025 proved the following uncertainty principle using Cauchy-Schwartz and Hölder's inequalities:

Additive Uncertainty Principle

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a nonzero signal with support in E, and let $\widehat{f}$ denote its Fourier transform with support in Σ.

$$N^d \leq |\Sigma| \cdot \Lambda_2^{1/3}(E)$$

At SMALL 2025, we investigated an application of **Gower's norms** to signal recovery and derived a significantly improved uncertainty bound.

Theorem(SMALL 2025)

$$N^d \leq |\Sigma| \left[\Lambda_2(E) - |E|^2\left[1 - \frac{N^d}{|E||\Sigma|}\right] - |E|(|E|-1)\left[1 - \sqrt{\frac{N^d}{|E||\Sigma|}}\sqrt{\frac{\Lambda_2(\Sigma)}{|\Sigma|^3}}\right]\right]^{1/3}$$

Intuition:
Additive energy measures structure, so lower energy means more randomness. Hence, a less structured support of f yields a stronger uncertainty principle.

This yields substantially better **signal recovery conditions** compared to previous bounds.

Theorem(SMALL 2025)

Suppose frequencies in $S \subseteq \mathbb{Z}_N^d$ are unobserved. Suppose $\Lambda_2(T) \leq |T|^\alpha$ where $2 \leq \alpha \leq 3, \forall T \subset \mathbb{Z}_N^d : |T| \leq 2|E|$, and also

$$|E|^3\Lambda_2(S) - |E|^3|S|(|S|-1)\left[1 - \frac{1}{(2|E|)^{(3-\alpha)/2}}\sqrt{\frac{N^d}{2|E||S|}}\right] - |E|^3|S|^2\left(1 - \frac{N^d}{2|E||S|}\right) < \frac{N^{3d}}{8}.$$

Then f can be recovered exactly and uniquely.

Future Directions

- Inspired by a fact that $\Lambda_1(E) = |E|^2$ we seek to find uncertainty principle that invokes a term
$$\Lambda_{k+1}(E) - \Lambda_k(E)(\dots)$$
To account number of non degenerate $k+1$ dimensional parallelepipeds.
- Consider other additive energy frames, such as number of tuples $a + b + c + d = e + f + g + h$, which naturally arises from Fourier Transform of
$$\sum_{x \in \mathbb{Z}_N^d} |\hat{f}(x)|^8.$$
- It is known that there are L_1 and L_2 minimisation algorithms for signal recovery. Is it possible to find U_k norm minimisation algorithm?

References

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Gower's U_k norm is defined by

$$\|f\|_{U_k}^{2^k} N^{(k+1)d} := \sum_{x, h_1, \dots h_k \in \mathbb{Z}_N^d} \prod_{w_j \in \{0,1\}} J^{w_1 + \dots w_k} f(x + w_1 h_1 + \dots w_k h_k).$$

In our project, we actively used U_2 norm, which simplifies to:

$$\|f\|_{U_2}^4 := N^{-3d} \sum_{x, h_1, h_2 \in \mathbb{Z}_N^d} \left[f(x)\overline{f(x+h_1)f(x+h_2)}f(x+h_1+h_2)\right].$$