Problems in the Theory of Low-Lying Zeros

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Simons Symposium on Families of Automorphic Forms and the Trace Formula, Rio Grande, Puerto Rico, Jan 27th, 2014

Introduction

I study n-level density of zeros of families of L-functions

n-level density: $\mathcal{F} = \cup \mathcal{F}_N$ a family of L-functions ordered by conductors, g_k an even Schwartz function: $D_{n,\mathcal{F}}(g) =$

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\frac{\log Q_f}{2\pi}\gamma_{j_1;f}\right) \cdots g_n\left(\frac{\log Q_f}{2\pi}\gamma_{j_n;f}\right)$$

As $N \to \infty$, n-level density converges to

$$\int g(\overrightarrow{x})\rho_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{x})d\overrightarrow{x} \;=\; \int \widehat{g}(\overrightarrow{u})\widehat{\rho}_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{u})d\overrightarrow{u}.$$

Conjecture (Katz-Sarnak)

(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

Results / Applications

- Results:
  - ◇ Agreement: Many families, small support.
  - ◇ Extending support: Related to arithmetic.

- Applications:
  - ◇ Class number: Bounds on growth rate.
  - ◇ Average rank: Vanishing at central point.

Techniques

- Explicit Formula: Convert sums over zeros to sums over Satake parameter moments.
- Averaging: Dirichlet, Petersson, Kuznetsov, ...
- Combinatorics: Showing agreement b/w NT and RMT.

Problem 1: Lower Order Terms

Explicit Formula

- π: cuspidal automorphic representation on GL_n.
- $Q_\pi > 0$: analytic conductor of $L(s,\pi) = \sum \lambda_\pi(n)/n^s$.
- By GRH the non-trivial zeros are $\frac{1}{2} + i\gamma_{\pi,j}$.
- Satake params $\{\alpha_{\pi,i}(p)\}_{i=1}^n$; $\lambda_\pi(p^\nu) = \sum_{i=1}^n \alpha_{\pi,i}(p)^\nu$.
- $L(s,\pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^n \left(1 - \alpha_{\pi,i}(p)p^{-s}\right)^{-1}$.

$$\sum_j g\left(\gamma_{\pi,j}\frac{\log Q_\pi}{2\pi}\right) = \widehat{g}(0) - 2\sum_{p,\nu} \widehat{g}\left(\frac{\nu \log p}{\log Q_\pi}\right) \frac{\lambda_\pi(p^\nu)\log p}{p^{\nu/2}\log Q_\pi}$$

1-Level Density

Assuming conductors constant in family $\mathcal{F}$, have to study

$$\begin{aligned}\lambda_f(p^\nu) &= \alpha_{f,1}(p)^\nu + \cdots + \alpha_{f,n}(p)^\nu \\ S_1(\mathcal{F}) &= -2\sum_p \hat{g}\left(\frac{\log p}{\log R}\right)\frac{\log p}{\sqrt{p}\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p)\right] \\ S_2(\mathcal{F}) &= -2\sum_p \hat{g}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2)\right]\end{aligned}$$

Corresponding classical compact group is determined by

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2) = c_{\mathcal{F}} = \begin{cases} 0 & \text{Unitary} \\ 1 & \text{Symplectic} \\ -1 & \text{Orthogonal.}\end{cases}$$

Open Problem: Lower order terms

Very similar to Central Limit Theorem.

- Universal behavior: main term controlled by first two moments of Satake parameters, agrees with RMT.
- First moment zero save for families of elliptic curves.
- Higher moments control convergence and can depend on arithmetic of family.

Open Problem:

Develop a theory of lower order terms to split the universality and see the arithmetic.

Problem 2:
Repulsion at the Central Point

Behavior of zeros near central point

For *one* L-function: good theory high up critical line.

For a *family* of L-functions: good theory as conductors tend to infinity.

Goal is to understand behavior **at central point** for finite conductors.

Questions (Elliptic Curve Families)

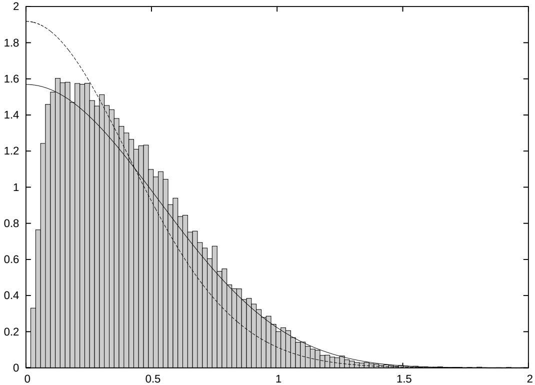
Excess rank: Expected vanishing at central point.

Repulsion: First zero above central point.

Open Problem

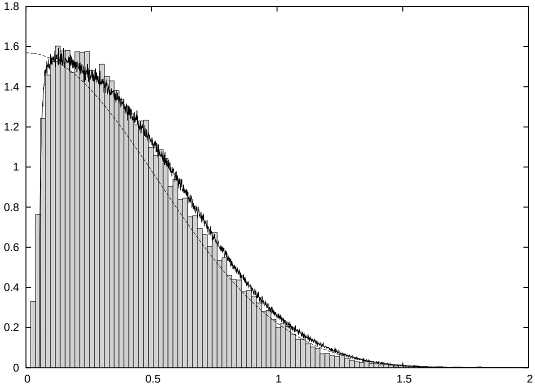
Model the observed behavior here (done) and extend to other families (in progress).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with N_{eff} (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): N_{eff} = 2 (solid) with discretisation, and $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

Problem 3: Combinatorics

Background

Different techniques to compute Number Theory and Random Matrix Theory.

Challenge is showing the two quantities are the same.

n-Level Density: Determinant Expansions from RMT

- U(N), $\mathrm{U}_k(N)$: $\det\left(K_0(x_j,x_k)\right)_{1\le j,k\le n}$
- USp(N): $\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\le n}$
- SO(even): $\det\left(K_1(x_j,x_k)\right)_{1\le j,k\le n}$
- SO(odd): $\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\le n} + \sum_{\nu=1}^{n}\delta(x_\nu)\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\ne\nu\le n}$

  where

$$K_\epsilon(x,y) = \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)} + \epsilon\frac{\sin\left(\pi(x+y)\right)}{\pi(x+y)}.$$

Alternative to Determinant Expansion

Expand Bessel-Kloosterman piece, use GRH to drop non-principal characters, change variables, main term is

$$\frac{b\sqrt{N}}{2\pi m}\int_0^\infty J_{k-1}(x)\widehat{\Phi_n}\left(\frac{2\log(bx\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}x}{\log R}$$

with $\Phi_n(x) = \phi(x)^n$.

Main Idea

Difficulty in comparison with classical RMT is that instead of having an n-dimensional integral of $\phi_1(x_1)\cdots\phi_n(x_n)$ we have a 1-dimensional integral of a new test function. This leads to harder combinatorics but allows us to appeal to the result from Iwaniec-Luo-Sarnak.

Problems

Open Problem:

Further develop alternatives to the Katz-Sarnak determinant expansions.

Open Problem:

Directly prove agreement for quadratic Dirichlet families (compare with Entin, Roddity-Gershon and Rudnick).

References

References to my work on these problems

- *Investigations of zeros near the central point of elliptic curve L-functions*, Experimental Mathematics **15** (2006), no. 3, 257–279. http://arxiv.org/pdf/math/0508150
- *Low lying zeros of L–functions with orthogonal symmetry* (with Christopher Hughes), Duke Mathematical Journal **136** (2007), no. 1, 115–172. http://arxiv.org/abs/math/0507450
- *Lower order terms in the 1-level density for families of holomorphic cuspidal newforms*, Acta Arithmetica **137** (2009), 51–98. http://arxiv.org/pdf/0704.0924.pdf
- *The effect of convolving families of L-functions on the underlying group symmetries* (with Eduardo Dueñez), Proceedings of the London Mathematical Society, 2009; doi: 10.1112/plms/pdp018. http://arxiv.org/pdf/math/0607688.pdf
- *The lowest eigenvalue of Jacobi Random Matrix Ensembles and Painlevé VI*, (with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith), Journal of Physics A: Mathematical and Theoretical **43** (2010) 405204 (27pp). http://arxiv.org/pdf/1005.1298
- *Models for zeros at the central point in families of elliptic curves* (with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith), J. Phys. A: Math. Theor. **45** (2012) 115207 (32pp). http://arxiv.org/pdf/1107.4426
- *The n-level density of zeros of quadratic Dirichlet L-functions* (with Jake Levinson), Acta Arithmetica **161** (2013), 145–182. http://arxiv.org/abs/1208.0930
- *Moment Formulas for Ensembles of Classical Compact Groups* (with Alan Chang, Geoffrey Iyer, Kyle Pratt, Nicholas Triantafillou and Minh-Tam Trinh), preprint.