Walking to infinity on the Fibonacci sequence

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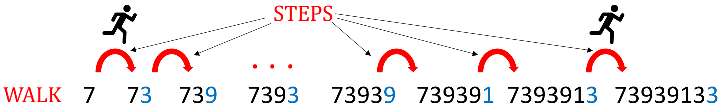
Introduction

Classical question

Is it possible to construct an infinite sequence on primes where each term is constructed by appending one digit to the right of the previous term?



- Starting with one digit, the above walk has the longest length possible.
- Starting with a prime of any digits – open.
- Appending a bounded number of digits to the right each time – open.

What about on the Fibonacci sequence?

Our question

Is it possible to construct an infinite sequence on the Fibonacci sequence where each term is constructed by appending one digit to the right of the previous term?

- The ratio between any two consecutive Fibonacci numbers is the golden ration, $1.618\ldots$. Hence, the Fibonacci sequence grows exponentially.
- By PNT, the n-th prime is asymptotic to $n \log n$, so primes grow much slower than Fibonacci numbers.
- Claim: It is impossible to construct such walks.

- Starting with one digit, there are only 5 possible walks as above.

Theorem statements

- Starting with a Fibonacci of any digits and appending one digit each time to the right:

Theorem 1

Given such conditions, it is impossible to walk to infinity. In particular, all such walks have a length of at most 2.

- Starting with a Fibonacci of N_0 digits and appending at most N digits each time to the right:

Theorem 2

If $N_0 \geq 2$, the length of the longest walk is then at most

$$\lfloor \log_2 \frac{N}{N_0 - 1} \rfloor + 2.$$

If $N_0 = 1$, the length of the longest walk is at most

$$\lfloor \log_2 \frac{N}{N_0} \rfloor + 2.$$

Lemma 1

Lemma 1

For all $m, k \in \mathbb{N}$, $\quad F_{k+1}F_m \;\leq\; F_{m+k} \;\leq\; F_{k+2}F_m.$

Proof.

Let $m \in \mathbb{N}$. Since the Fibonacci sequence is increasing, we have

$$F_m \leq F_{m+1} \leq 2F_m \;\rightarrow\; 2F_m \leq F_{m+2} \leq 3F_m.$$

Now, suppose that for all k with $2 \leq k \leq r$, $F_{k+1}F_m \leq F_{m+k} \leq F_k F_m$. Taking $k = r - 1$ and r gives us

$$F_r F_m \;\leq\; F_{m+r-1} \;\leq\; F_{r+1}F_m$$

$$F_{r+1}F_m \;\leq\; F_{m+r} \;\leq\; F_{r+2}F_m$$

respectively.

$$F_{r+2}F_m \;\leq\; F_{m+r+1} \;\leq\; F_{r+3}F_m.$$

□

Lemma 2

Lemma 2

For all $m > k \in \mathbb{N}, k > 2$,

$$F_{m+k} = (F_{k+2} - F_{k-2})F_m + (-1)^{k+1}F_{m-k}.$$

Proof.

Let $\varphi = (1+\sqrt{5})/2$, the golden ratio. Using the supposition, we compute that

$$\varphi^4 - 1 = \sqrt{5}\varphi^2.$$

Moreover, Binet's formula gives

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}.$$

The statement can be derived by direct computation along with the above facts. □

Theorem 1

Theorem 1

It is impossible to construct an infinite walk on the Fibonacci sequence by appending exactly one digit at a time to the right.

Proof.

Starting $F_m \geq 1$, if we append $d \in \{0, 1, 2, \ldots, 9\}$, the newly appended number is $10F_m + d$.

$$5F_m \leq F_{m+4} \leq 8F_m \leq F_{m+5} \leq 13F_m \leq F_{m+6} \leq 21F_m, \qquad (1)$$

$$\text{For all, } m > 5 \in \mathbb{N}, \; F_{m+5} = 11F_m + F_{m-5}. \qquad (2)$$

Since $8F_m + d \leq 10F_m + d \leq 13F_m + d$, (1) implies that $10F_m + d$ is either F_{m+5} or F_{m+6}.

- $10F_m + d = F_{m+6}$: We have that $10F_m + d \geq 13F_m$ and, thus, $d \geq 3F_m$. Since d is a single-digit number, F_m can be 1, 2, or 3.
- $10F_m + d = F_{m+5}$: If $m > 5$, (2) implies $10F_m + d = 11F_m + F_{m-5}$, so $F_m = 8$. Otherwise if $m \leq 5$, F_m is either $1, 2, 3, 5$, or 8.

□

Theorem statements

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Theorem 1

Given such conditions, it is impossible to walk to infinity. In particular, all such walks have a length of at most 2.

- Starting with a Fibonacci of N_0 digits and appending at most N digits each time to the right:

Theorem 2

If $N_0 \geq 2$, the length of the longest walk is then at most

$$\lfloor \log_2 \frac{N}{N_0 - 1} \rfloor + 2.$$

If $N_0 = 1$, the length of the longest walk is at most

$$\lfloor \log_2 \frac{N}{N_0} \rfloor + 2.$$

Lemma 3

Lemma 3

For all $N \in \mathbb{N}$, there exists no natural number $k \geq 2$ such that

$$F_{k+2} - F_{k-2} = 10^N.$$

Proof.

This is enough to show that there is no k such that $F_{k+2} \equiv F_{k-2} \pmod{10}$.

By observing the first 100 Fibonacci numbers, we notice that $F_{61} \equiv F_1 \pmod{10}$ and $F_{62} \equiv F_2 \pmod{10}$. Then, for $n \in \mathbb{N}$, $F_{60+n} \equiv F_n \pmod{10}$.

The periodic property tells us that if there exists no pair of Fibonacci numbers F_{k+2} and F_{k-2}, where $2 \leq k \leq 62$, such that $F_{k+2} - F_{k-2} \equiv 0 \pmod{10}$, then there is no $k \in \mathbb{N}$ such that $F_{k+2} - F_{k-2} \equiv 0 \pmod{10}$; as a result, it is impossible to have $F_{k+2} - F_{k-2} = 10^N$.

Going through the list of Fibonacci numbers again, we find no such pair in the first 62 Fibonacci numbers, which completes our proof. □

Lemma 4

Lemma 4

It is impossible to construct an infinite walk on the Fibonacci sequence by appending exactly N digits at a time, where N is a fixed positive integer. In particular, any appendable number in the walk must be at most $8/7 \cdot (10^N - 1)$.

Proof.

Let N be a fixed positive integer and F_m be the starting number of a walk. Similar to Theorem 1, the next number in the walk can be written as $10^N F_m + d$, where $0 \leq d \leq 10^N - 1$.
Let k be such that $F_{k+1} \leq 10^N$ and $F_{k+2} > 10^N$. By Lemma 1, we have

$$F_{m+k-1} \leq F_{k+1}F_m \leq F_{m+k} \leq F_{k+2}F_m \leq F_{m+k+1}. \qquad (3)$$

There are 3 possible cases:
(I) $10^N F_m + d = F_{m+k}$,
(II) $10^N F_m + d < F_{k+1}F_m$, and
(III) $10^N F_m + d > F_{k+2}F_m$.

Corollaries of Lemma 4

Corollary 1

The implication of Lemma 4 is that any appendable number in a walk must contain at most

$$\lfloor \log(8/7 \cdot (10^N - 1)) + 1 \rfloor = \lfloor 0.058 + N + 1 \rfloor = N + 1$$

digits, given we append exactly N digits each time. Since any number not greater than $8/7 \cdot (10^N - 1)$ will contain at least $N + 1$ digits after appended by N digits one time, we can append at most twice.

Corollary 2

If a Fibonacci number has at least $N + 2$ digits, we cannot append $1, 2, \ldots,$ or N digits to the right of that number to obtain another Fibonacci number because it already has too many digits.

Theorem 2

Theorem 2

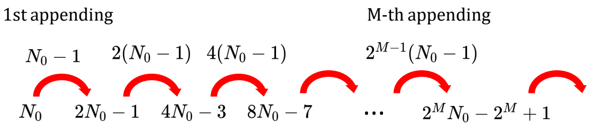
Given we append at most N digits to the right each time and the starting number contains $N_0 \geq 2$ digits, the length of the longest walk is then at most $\lfloor \log_2 \frac{N}{N_0 - 1} \rfloor + 2$. If $N_0 = 1$, the length of the longest walk is at most $\lfloor \log_2 \frac{N}{N_0} \rfloor + 2$.

Proof.

Start with a Fibonacci number A_0 that has N_0 digits. Corollary 2 implies that we cannot append $1, \ldots, N_0 - 2$ digits to A_0. Thus, we can only append $N_0 - 1$ or more digits in the first appending.

Then, after the first appending, the new number A_1 contains at least $N_0 + N_0 - 1 = 2N_0 - 1$ digits. Again, Corollary 2 implies that we can only append $2N_0 - 2$ or above number of digits in the second appending.

Theorem 2



Proof.

Repeating the process above, we need to append at least $2^{M-1}(N_0 - 1)$ digits at the M-th step. Hence, we determine the largest M as follows:

$$\begin{aligned} 2^{M-1}(N_0 - 1) &\leq N \\ M &\leq \log_2 \frac{N}{N_0 - 1} + 1. \end{aligned}$$

Therefore, the length of the longest walk is at most $\lfloor \log_2 \frac{N}{N_0 - 1} \rfloor + 2$, including the starting number.

This formula does not work for $N_0 = 1$ since we do not want to append $N_0 - 1 = 0$ digit. However, by similar analysis, we obtain that $\lfloor \log_2 \frac{N}{N_0} \rfloor + 2$ is the length of the longest walk starting with a single digit. □

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