Solutions to a Pair of Diophantine Equations

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https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

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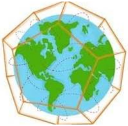
Polymath Junior Program



- **Provide research opportunities to undergraduates.**
- **Online, runs in the spirit of the Polymath Project.**
- **Projects run by researcher with experience in undergraduate mentoring.**
- **Most 15-25 students, a main mentor, grad students / postdocs assisting.**

https://geometrynyc.wixsite.com/polymathreu

Outline

- Introduction and results from Polymath Jr. 24
- What we did in Polymath Jr. 25

Introduction

A pair of equations

For relatively prime $a, b \in \mathbb{N}$, consider

$$\begin{aligned} ax + by &= \frac{(a-1)(b-1)}{2} \text{ and} \\ 1 + ax + by &= \frac{(a-1)(b-1)}{2}. \end{aligned}$$

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Theorem (Beiter (1964), extended by Chu (2020))

Exactly one of the equations has a nonnegative integral solution (x, y). The solution is unique.

Fibonacci numbers

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$$a\,\boxed{x} + b\,\boxed{y} = \frac{(a-1)(b-1)}{2}$$

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$$F_n \boxed{x} + F_{n+1} \boxed{y} \ = \ \frac{(F_n - 1)(F_{n+1} - 1)}{2}$$

$$1 + F_n \boxed{x} + F_{n+1} \boxed{y} \ = \ \frac{(F_n - 1)(F_{n+1} - 1)}{2}$$

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$$(x, y) = ?$$

Chu's six cases (2020)

$$F_{6k} \cdot \boxed{\frac{F_{6k-1}-1}{2}} + F_{6k+1} \cdot \boxed{\frac{F_{6k-1}-1}{2}} = \frac{(F_{6k}-1)(F_{6k+1}-1)}{2}$$

$$F_{6k+1} \cdot \boxed{\frac{F_{6k+1}-1}{2}} + F_{6k+2} \cdot \boxed{\frac{F_{6k-1}-1}{2}} = \frac{(F_{6k+1}-1)(F_{6k+2}-1)}{2}$$

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$$1 + F_{6k+5} \cdot \boxed{\frac{F_{6k+4}-1}{2}} + F_{6k+6} \cdot \boxed{\frac{F_{6k+4}-1}{2}} = \frac{(F_{6k+5}-1)(F_{6k+6}-1)}{2}$$

Fibonacci Squared (Polymath Jr. 2024)

If $n \equiv 0, 2, 3, 5 \mod 6$,

$$F_n^2 \cdot \boxed{\left(F_n^2 - \frac{F_{n-1}^2 + 1}{2}\right)} + F_{n+1}^2 \cdot \boxed{\frac{F_{n-1}^2 - 1}{2}} = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}.$$

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If $n \equiv 4 \mod 6$,

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Fibonacci Cubed (Polymath Jr. 2024)

For $n \geq 2$,

$$F_{2n-1}^3 \cdot \boxed{\sum_{i=1}^{2n-1} (-1)^{i-1} F_i^3} + F_{2n}^3 \cdot \boxed{\sum_{i=2}^{2n-2} F_i^3} = \frac{(F_{2n-1}^3 - 1)(F_{2n}^3 - 1)}{2};$$

$$1 + F_{2n}^3 \cdot \boxed{\left(\sum_{i=1}^{2n} (-1)^i F_i^3 - 1\right)} + F_{2n+1}^3 \cdot \boxed{\sum_{i=2}^{2n-1} F_i^3} = \frac{(F_{2n}^3 - 1)(F_{2n+1}^3 - 1)}{2}.$$

Problem 1

For $(i, j) \in \mathbb{N}^2$, find the nonnegative integral solution (x, y) to

$$F_n^i \cdot x + F_{n+1}^j \cdot y = \frac{(F_n^i - 1)(F_{n+1}^j - 1)}{2}$$

or

$$1 + F_n^i \cdot x + F_{n+1}^j \cdot y = \frac{(F_n^i - 1)(F_{n+1}^j - 1)}{2}.$$

What we did in Polymath Jr. 25

Fibonacci-like sequences

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$$t_n^{(u,v)}x + t_{n+1}^{(u,v)}y = \frac{(t_n^{(u,v)} - 1)(t_{n+1}^{(u,v)} - 1)}{2}$$

$$t_n^{(u,v)}x + t_{n+1}^{(u,v)}y + 1 = \frac{(t_n^{(u,v)} - 1)(t_{n+1}^{(u,v)} - 1)}{2}$$

Nonnegative integral $(x, y) = ?$

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Nonnegative integral (x, y) $=$? Depend on n modulo 6.

The solution (x, y)

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Fibonacci pairs (3): $(F_{6n}, F_{6n+3}), (F_{6n+1}, F_{6n+4}), (F_{6n+2}, F_{6n+5})$

The solution (x, y)

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Fibonacci pairs (3): $(F_{6n}, F_{6n+3}), (F_{6n+1}, F_{6n+4}), (F_{6n+2}, F_{6n+5})$

In each case, the solution (x, y) depends further on (u, v).

Sample case: $n \equiv 4 \mod 6$

Polymath Jr. 25 Diophantine Group

Given $(u, v, n, r) \in \mathbb{Z}^4$ with even n, it holds that

$$1 + \frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v+1}{u}F_n - 1\right)t_n^{(u,v)} + \frac{1}{2}\left(rF_{n-2} + \frac{vr-1}{u}F_{n-1} - 1\right)t_{n+1}^{(u,v)} = \frac{(t_n^{(u,v)}-1)(t_{n+1}^{(u,v)}-1)}{2}$$

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and

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For even n,

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Proof:

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Proof:

$$t_n^{(u,v)} = F_{n-2}u + F_{n-1}v$$

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Proof:

$$t_n^{(u,v)} = F_{n-2}u + F_{n-1}v$$

$$F_{n-1}F_{n+1} - F_n^2 = 1 \text{ (for even } n)$$

Sample case: $n \equiv 4 \mod 6$

$$1 + \frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v+1}{u}F_n - 1\right)t_n^{(u,v)} +$$
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$$= \frac{1}{2} + \frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v+1}{u}F_n\right)(F_{n-2}u + F_{n-1}v) + \frac{1}{2}\left(rF_{n-2} + \frac{vr-1}{u}F_{n-1}\right)(F_{n-1}u + F_n v) - \frac{1}{2}(t_n^{(u,v)} + t_{n+1}^{(u,v)} - 1)$$

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$$\frac{1}{2}\left(rF_{n-2}+\frac{vr-1}{u}F_{n-1}-1\right)t_{n+1}^{(u,v)}$$

$$=\frac{1}{2}+\frac{1}{2}\left((u-r)F_{n-1}+\frac{(u-r)v+1}{u}F_n\right)(F_{n-2}u+F_{n-1}v)+$$
$$\frac{1}{2}\left(rF_{n-2}+\frac{vr-1}{u}F_{n-1}\right)(F_{n-1}u+F_nv)-\frac{1}{2}(t_n^{(u,v)}+t_{n+1}^{(u,v)}-1)$$

$$=\frac{1}{2}\left(1+F_{n-2}F_n-F_{n-1}^2\right)+\frac{1}{2}\underbrace{(uF_{n-2}+vF_{n-1})}_{t_n^{(u,v)}}\underbrace{(uF_{n-1}+vF_n)}_{t_{n+1}^{(u,v)}}-\frac{1}{2}(t_n^{(u,v)}+t_{n+1}^{(u,v)}-1)$$

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$$= \frac{1}{2}(t_n^{(u,v)} - 1)(t_{n+1}^{(u,v)} - 1)$$

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Polymath Jr. 25 Diophantine Group

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$$\begin{cases} x &= \frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v+1}{u}F_n - 1\right) \\ y &= \frac{1}{2}\left(rF_{n-2} + \frac{vr-1}{u}F_{n-1} - 1\right) \end{cases} ?$$

NOT YET!

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NOT YET!

Need r such that the boxed are nonnegative integers

Choose r when $n \equiv 4 \mod 6$

Lemma

Given $(u, v) \in \mathbb{N}^2$ with $\gcd(u, v) = 1$ and odd u,

$$\exists! \text{ odd } \boldsymbol{r} \in [1, u] \text{ with } v\boldsymbol{r} \equiv \pm 1 \mod u.$$

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Denote $\boldsymbol{r}$ by $\mathbb{O}(u, v)$.

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Assume $u \geq 3$.

$\gcd(u, v) = 1 \implies \{1 \cdot v, 2 \cdot v, \ldots, u \cdot v\}$ is a complete modulo system of u.

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$\exists x_1, x_2 \in [1, u-1]$ s.t. $vx_1 \equiv 1 \mod u$ and $vx_2 \equiv -1 \mod u$.

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Lemma

Given $(u, v) \in \mathbb{N}^2$ with $\gcd(u, v) = 1$ and odd u,

$$\exists! \text{ odd } \boldsymbol{r} \in [1, u] \text{ with } v\boldsymbol{r} \equiv \pm 1 \mod u.$$

Assume $u \geq 3$.

$\gcd(u, v) = 1 \implies \{1 \cdot v, 2 \cdot v, \ldots, u \cdot v\}$ is a complete modulo system of u.

$\exists x_1, x_2 \in [1, u-1]$ s.t. $vx_1 \equiv 1 \mod u$ and $vx_2 \equiv -1 \mod u$.

$\implies u \mid v(x_1 + x_2) \implies u | (x_1 + x_2) \implies x_1 + x_2 = u.$

Solutions when $n \equiv 4 \mod 6$

If u is odd and $v\mathbb{O}(u,v) \equiv 1 \mod u$ or u is even and $v\mathbb{O}(u,v) \equiv 1 \mod 2u$,

$$1 + t_n^{(u,v)} \cdot \boxed{\frac{1}{2}\left((u - \mathbb{O}(u,v))F_{n-1} + \frac{(u - \mathbb{O}(u,v))v + 1}{u}F_n - 1\right)} +$$

$$t_{n+1}^{(u,v)} \cdot \boxed{\frac{1}{2}\left(\mathbb{O}(u,v)F_{n-2} + \frac{v\mathbb{O}(u,v) - 1}{u}F_{n-1} - 1\right)} = \frac{(t_n^{(u,v)} - 1)(t_{n+1}^{(u,v)} - 1)}{2}.$$

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If u is odd, $u \geq 3$, and $v\mathbb{O}(u,v) \equiv -1 \mod u$ or u is even and $v\mathbb{O}(u,v) \equiv -1 \mod 2u$,

$$t_n^{(u,v)} \cdot \boxed{\frac{1}{2}\left((u-\mathbb{O}(u,v))F_{n-1} + \frac{(u-\mathbb{O}(u,v))v-1}{u}F_n - 1\right)} +$$

$$t_{n+1}^{(u,v)} \cdot \boxed{\frac{1}{2}\left(\mathbb{O}(u,v)F_{n-2} + \frac{v\mathbb{O}(u,v)+1}{u}F_{n-1} - 1\right)} = \frac{(t_n^{(u,v)}-1)(t_{n+1}^{(u,v)}-1)}{2}.$$

Nonnegative, integral solutions for $n \equiv 4 \mod 6$

u is odd and $v\mathbb{O}(u,v) \equiv 1 \mod u$:

$$1 + t_n^{(u,v)} \cdot \frac{1}{2}\left((u - \mathbb{O}(u,v)) \underbrace{F_{n-1}}_{even} + \underbrace{\frac{(u-\mathbb{O}(u,v))v+1}{u}}_{odd} \underbrace{F_n}_{odd} - 1 \right) +$$

$$t_{n+1}^{(u,v)} \cdot \frac{1}{2}\left(\underbrace{\mathbb{O}(u,v)}_{odd} \underbrace{F_{n-2}}_{odd} + \frac{v\mathbb{O}(u,v)-1}{u} \underbrace{F_{n-1}}_{even} - 1 \right) = \frac{(t_n^{(u,v)}-1)(t_{n+1}^{(u,v)}-1)}{2}$$

Application: $u = v = 1$ (Fibonacci) and $n = 6k + 4$

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$$1 + F_{6k+4} \cdot \frac{1}{2} \underbrace{\left((u - \mathbb{O}(u, v))F_{6k+3} + \frac{(u - \mathbb{O}(u, v))v + 1}{u} F_{6k+4} - 1 \right)}_{F_{6k+4}-1} +$$

$$F_{6k+5} \cdot \frac{1}{2} \underbrace{\left(\mathbb{O}(u, v)F_{6k+2} + \frac{v\mathbb{O}(u, v) - 1}{u} F_{6k+3} - 1 \right)}_{F_{6k+2}-1} = \frac{(F_{6k+4} - 1)(F_{6k+5} - 1)}{2}$$

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This matches Chu's (2020) ✓ :

$$1 + F_{6k+4} \cdot \boxed{\frac{F_{6k+4} - 1}{2}} + F_{6k+5} \cdot \boxed{\frac{F_{6k+2} - 1}{2}} = \frac{(F_{6k+4} - 1)(F_{6k+5} - 1)}{2}$$

Problem 2

Find the formula for the solutions (x, y) to

$$a \cdot x + b \cdot y = \frac{(a-1)(b-1)}{2}$$

or

$$1 + a \cdot x + b \cdot y = \frac{(a-1)(b-1)}{2},$$

where a and b are taken from other recursively defined sequences.

Future investigation

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Thank you!