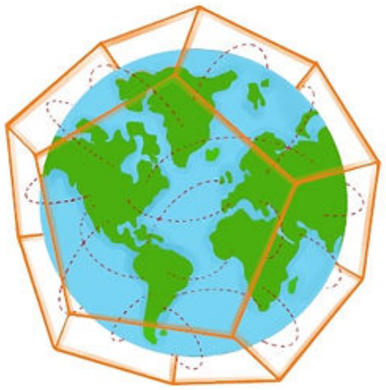
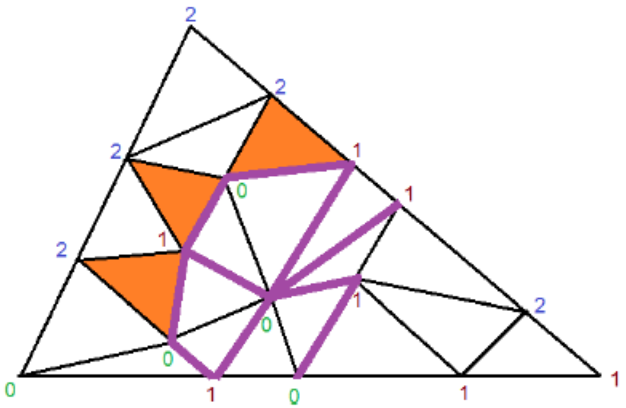
From Sperner’s Lemma to Spurring Research





Steven J. Miller (sjm1@williams.edu)

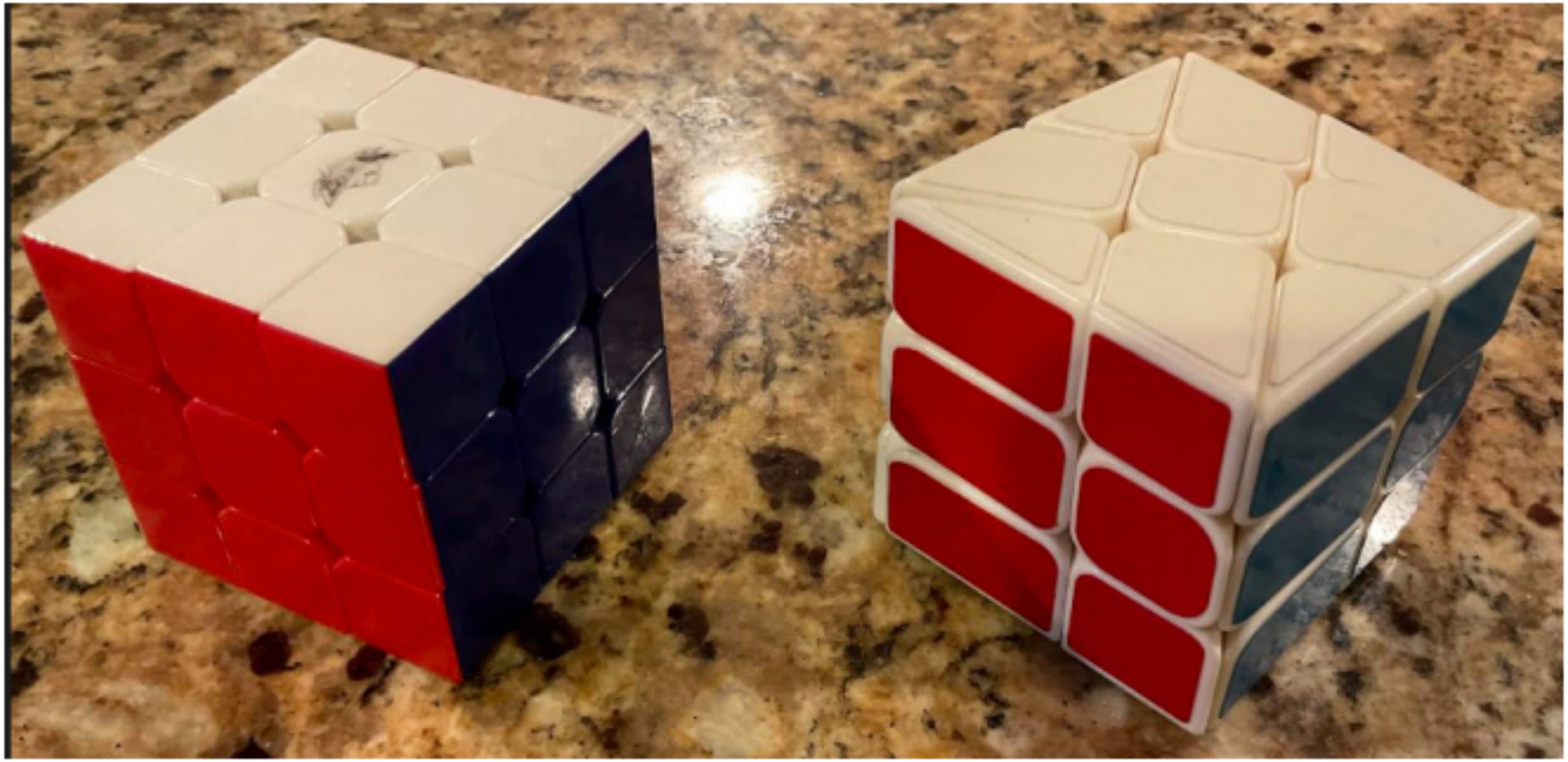
Morse ‘96, Williams College and the Fibonacci Association

https://web.williams.edu/Mathematics/sjmiller/public_html/

Yale University, July 23, 2025

QUESTION:

How do you prove something exists?



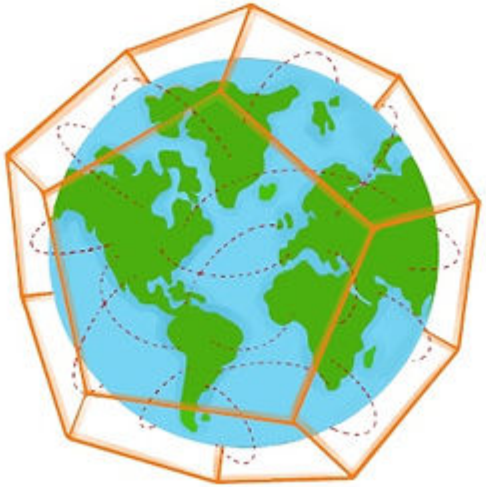
QUESTION:

How do you prove something exists?

GOALS

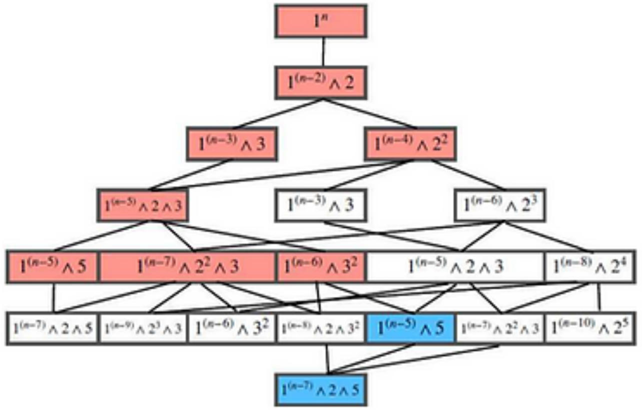
- **Discuss number theory existence proofs**

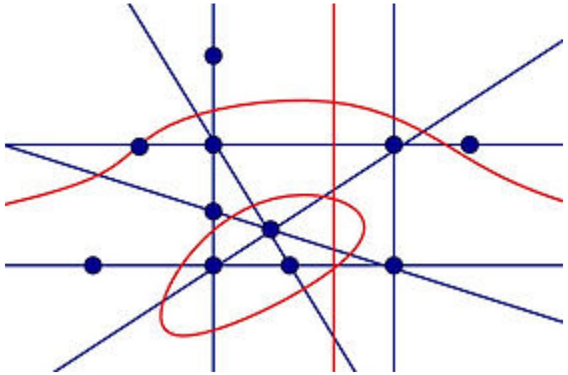
- **Invitation to summer research: Polymath JR REU**
  geometrynyc.wixsite.com/polymathreu

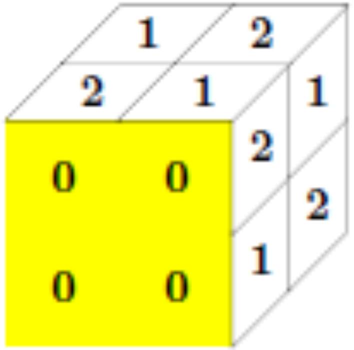


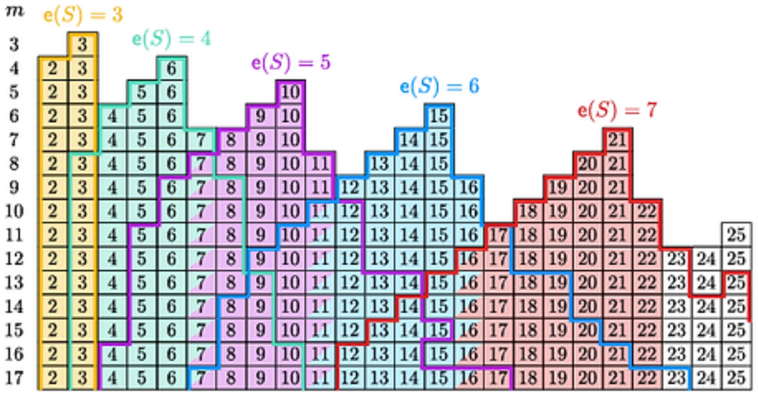
Our goal is to provide research opportunities to every undergraduate who wishes to explore advanced mathematics. This online program consists of research projects in a variety of mathematical topics and runs in the spirit of the Polymath Project. Each project is mentored by an active researcher with experience in undergraduate mentoring.

Each project consists of 15-25 undergraduates, a main mentor, and graduate students and postdocs as additional mentors. The group works towards solving a research problem and writing a paper. Each participant decides what they wish to obtain from the program, participating accordingly. The program is partially supported by NSF award DMS-2218374.









CONSTRUCTIVE

- $ax + b = 0$, root $x = -b/a$.
- $ax^2 + bx + c = 0$, roots $(-b \pm \sqrt{b^2 - 4ac})/2a$.

Cubic: For fun, here's the solution to $ax^3 + bx^2 + cx + d = 0$

```
Solve[a x^3 + b x^2 + c x + d == 0, x]
```

$$\left\{\left\{x \to -\frac{b}{3a} - \frac{2^{1/3}\left(-b^2+3ac\right)}{3a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} + \frac{\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{3\times 2^{1/3}a}\right\},\right.$$

$$\left\{x \to -\frac{b}{3a} + \frac{\left(1+i\sqrt{3}\right)\left(-b^2+3ac\right)}{3\times 2^{2/3}a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} - \frac{\left(1-i\sqrt{3}\right)\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{6\times 2^{1/3}a}\right\},$$

$$\left.\left\{x \to -\frac{b}{3a} + \frac{\left(1-i\sqrt{3}\right)\left(-b^2+3ac\right)}{3\times 2^{2/3}a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} - \frac{\left(1+i\sqrt{3}\right)\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{6\times 2^{1/3}a}\right\}\right\}$$

One of the solutions to quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$

```
Solve[a x^4 + b x^3 + c x^2 + d x + e == 0, x]
```

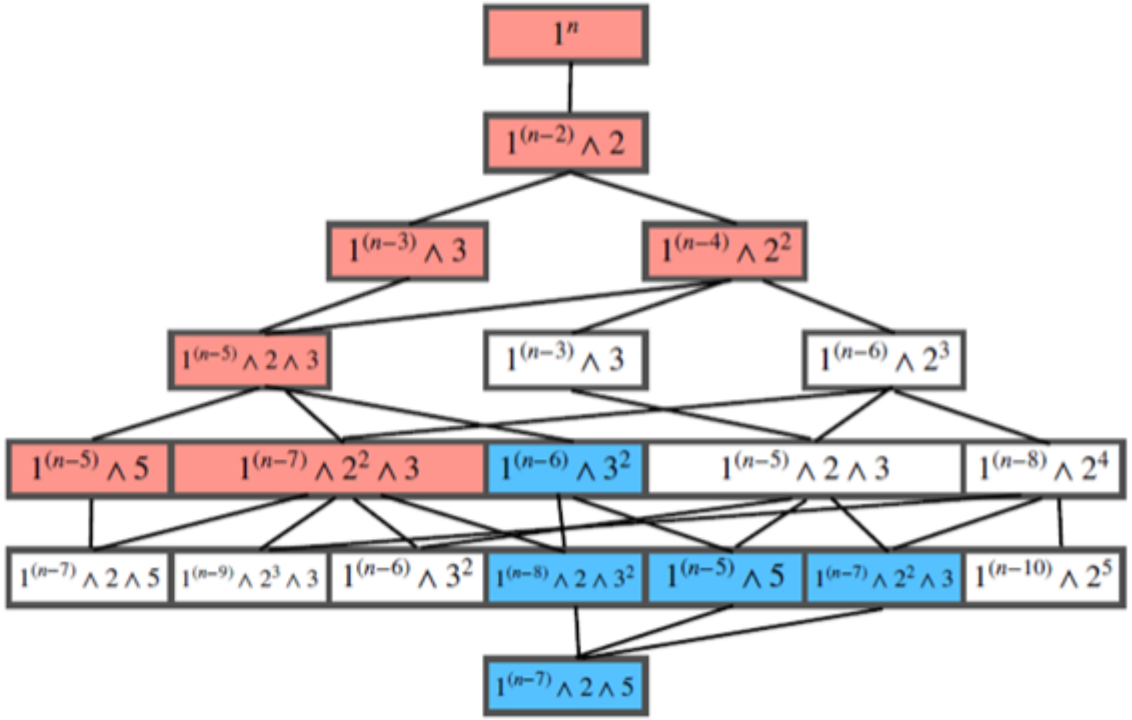
$$\Bigg\{\Bigg\{x \to -\frac{b}{4a} - \frac{1}{2}\sqrt{\left(\frac{b^2}{4a^2} - \frac{2c}{3a} + \left(2^{1/3}\left(c^2 - 3bd + 12ae\right)\right)\Big/\left(3a\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\right) + \frac{1}{3 \times 2^{1/3}a}\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\right)} - \frac{1}{2}\sqrt{\left(\frac{b^2}{2a^2} - \frac{4c}{3a} - \left(2^{1/3}\left(c^2 - 3bd + 12ae\right)\right)\Big/\left(3a\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\right) - \frac{1}{3 \times 2^{1/3}a}\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3} - \left(-\frac{b^3}{a^3} + \frac{4bc}{a^2} - \frac{8d}{a}\right)\Big/\left(4\sqrt{\left(\frac{b^2}{4a^2} - \frac{2c}{3a} + \left(2^{1/3}\left(c^2 - 3bd + 12ae\right)\right)\Big/\left(3a\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\right) + \frac{1}{3 \times 2^{1/3}a}\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\right)}\right)\right)}\Bigg\},$$

Zeckendorf Decompositions and Games

• From Monovariants to Zeckendorf Decompositions and Games, and Random Matrix Theory, Williams College (7/14/21) and Texas Tech (7/29/21). pdf (video: https://youtu.be/Kayru_V75V8)

Player One won in 9 moves.

10	0	0	0	0
8	1	0	0	0
6	2	0	0	0
7	0	1	0	0
5	1	1	0	0
5	0	0	1	0
3	1	0	1	0
1	2	0	1	0
0	1	1	1	0
0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$



•Zeckendorf Games

- •The Zeckendorf Game (with Paul Baird-Smith, Alyssa Epstein and Kristen Flint), to appear in the Proceedings of CANT 2018. pdf
- •The Generalized Zeckendorf Game (with Paul Baird-Smith, Alyssa Epstein and Kristen Flint), Fibonacci Quarterly (**57** (2019) no. 5, 1-14) pdf
- •Extending Zeckendorf's Theorem to a Non-constant Recurrence and the Zeckendorf Game on this Non-constant Recurrence Relation (with Ela Boldyriew, Anna Cusenza, Linglong Dai, Pei Ding, Aidan Dunkelberg, John Haviland, Kate Huffman, Dianhui Ke, Daniel Kleber, Jason Kuretski, John Lentfer, Tianhao Luo, Clayton Mizgerd, Vashisth Tiwari, Jingkai Ye, Yunhao Zhang, Xiaoyan Zheng, and Weiduo Zhu), Fibonacci Quarterly. (**5** (2020), 55-76**)** pdf
- •Deterministic Zeckendorf Games (with Ruoci Li, Xiaonan Li, Clay Mizgerd, Chenyang Sun, Dong Xia, And Zhyi Zhou), Fibonacci Quarterly. (**58** (2020), no. 5, 152-160**)** p3df
- •Winning Strategy for the Multiplayer and Multialliance Zeckendorf Games (with Anna Cusenza, Aiden Dunkelberg, Kate Huffman, Dianhui Ke, Daniel Kleber, Clayton Mizgerd, Vashisth Tiwari, Jingkai Ye, Xiaoyan Zheng), Fibonacci Quarterly. (**59** (2021), 308--318**)** pdf
- •Bounds on Zeckendorf Games (with Anna Cusenza, Aiden Dunkelberg, Kate Huffman, Dianhui Ke, Daniel Kleber, Micah McClatchey, Clayton Mizgerd, Vashisth Tiwari, Jingkai Ye, Xiaoyan Zheng), Fibonacci Quarterly. pdf (**1** (2022), no. 1, 57--71**)**
- •Winning Strategies for the Generalized Zeckendorf Game (Steven J. Miller, Eliel Sosis, Jingkai Ye), Fibonacci Quarterly. (Conference Proceedings: 20th International Fibonacci Conference: **60** (2022), no.5, 270--292). pdf
- •The Accelerated Zeckendorf Game (Diego Garcia-Fernandezsesma, Steven J. Miller, Thomas Rascon, Risa Vandegrift, and Ajmain Yamin), to appear in the Fibonacci Quarterly. pdf
- •Towards the Gaussianity of Random Zeckendorf Games (Justin Cheigh, Guilherme Zeus Dantas E Moura, Ryan Jeong, Jacob Lehmann Duke, Wyatt Milgrim, Steven J. Miller, and Prakod Ngamlamai), submitted to the CANT Proceedings. pdf

https://press.princeton.edu/books/hardcover/9780691120607/an-invitation-to-modern-number-theory

Existence by Infinity: Primes in Progression

Theorem 2.3.5 (Dirichlet's Theorem on Primes in Arithmetic Progressions). *Let a and m be relatively prime integers. Then there are infinitely many primes in the progression $nm + a$. Further, for a fixed m to first order all relatively prime a give progressions having the same number of primes. This means that if $\pi_{m,b}(x)$ denotes the number of primes at most x congruent to b modulo m then $\lim_{x\to\infty} \frac{\pi_{m,a}(x)}{\pi_{m,b}(x)} = 1$ for all a, b relatively prime to m. As there are $\phi(m)$ numbers relatively prime to m and $\pi(x)$ primes at most x, we have that as $x \to \infty$*

$$\pi_{m,a}(x) \;=\; \frac{\pi(x)}{\phi(m)} + \text{lower order terms.} \tag{2.46}$$

Composites in Progression

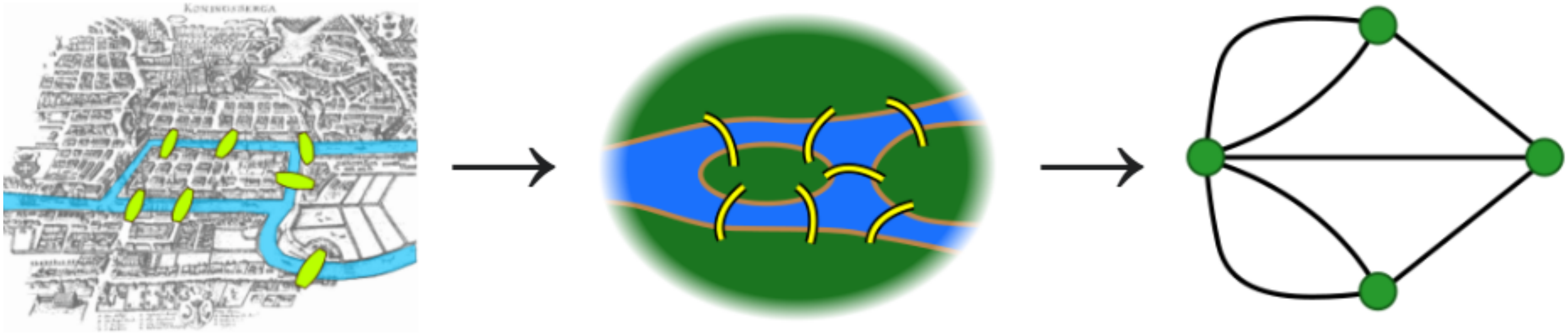
Theorem (Telhcirid): Let a and m be any pair of positive integers. Then there are infinitely many composites congruent to a mod m.

(Composite Numbers in Arithmetic Progression, joint with H. V. Chu and J. Siktar, to appear in Mathematics Magazine: https://arxiv.org/pdf/2411.03330)

If $a > 1$ look at $n \cdot am + a$

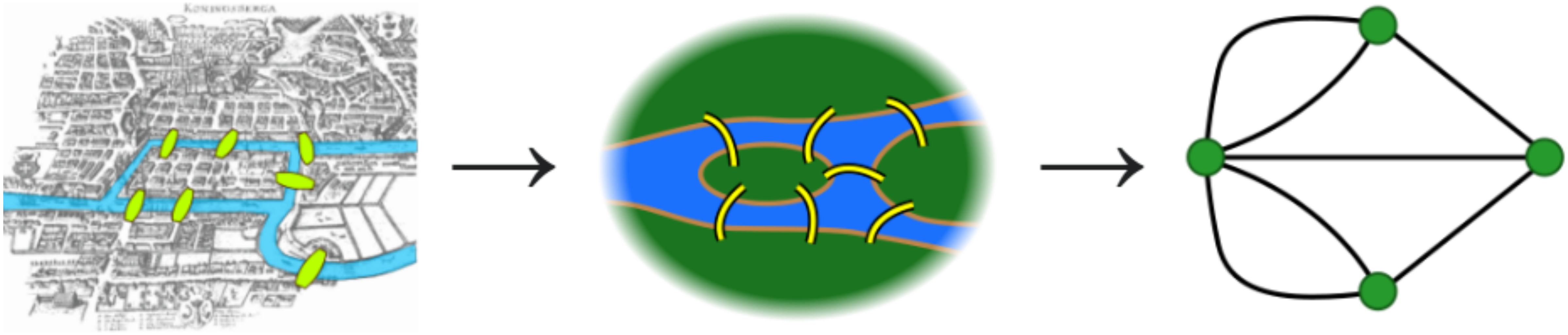
$$\pi(x) \sim \frac{x}{\log x}$$

Existence by Parity: Seven Bridges of Königsberg



The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands—Kneiphof and Lomse—which were connected to each other, and to the two mainland portions of the city, by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.
https://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg

Existence by Parity: Seven Bridges of Königsberg



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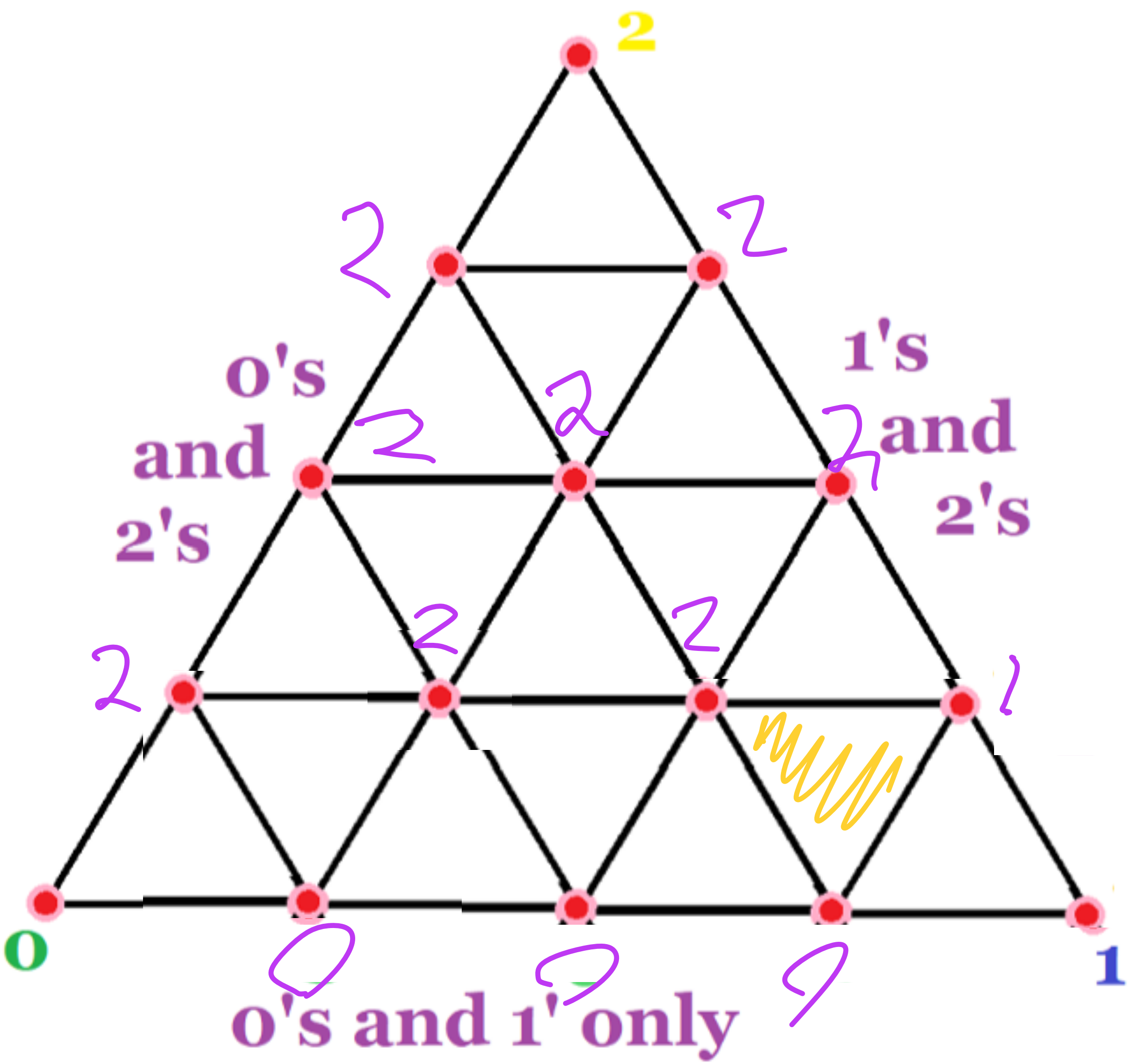
https://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg

Parity:
Each vertex odd degree.
Except for initial and final vertex all others hit even number of times.
All degrees odd so impossible….

The Sperner (or Triangle) Game:

Bank
6

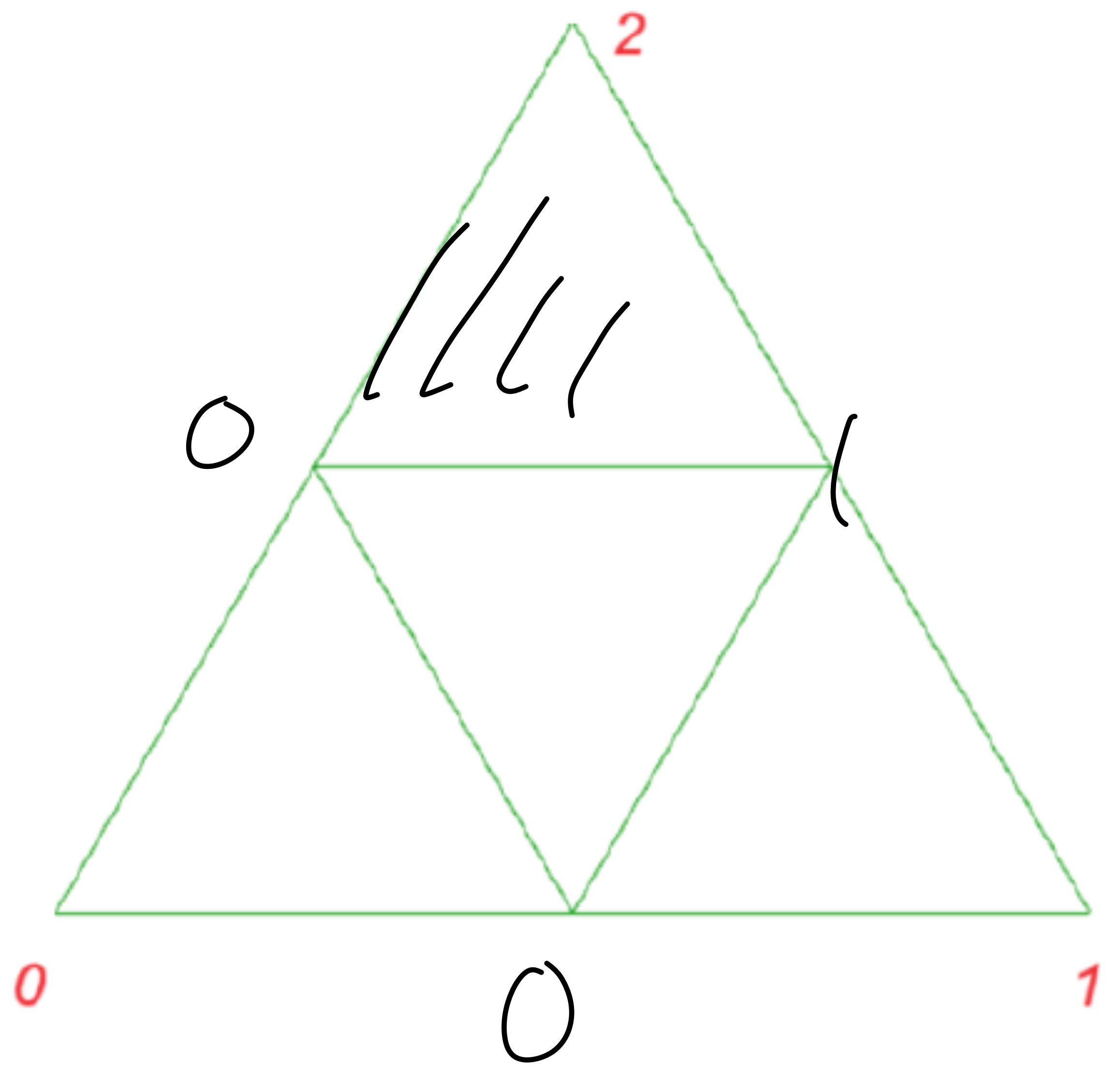


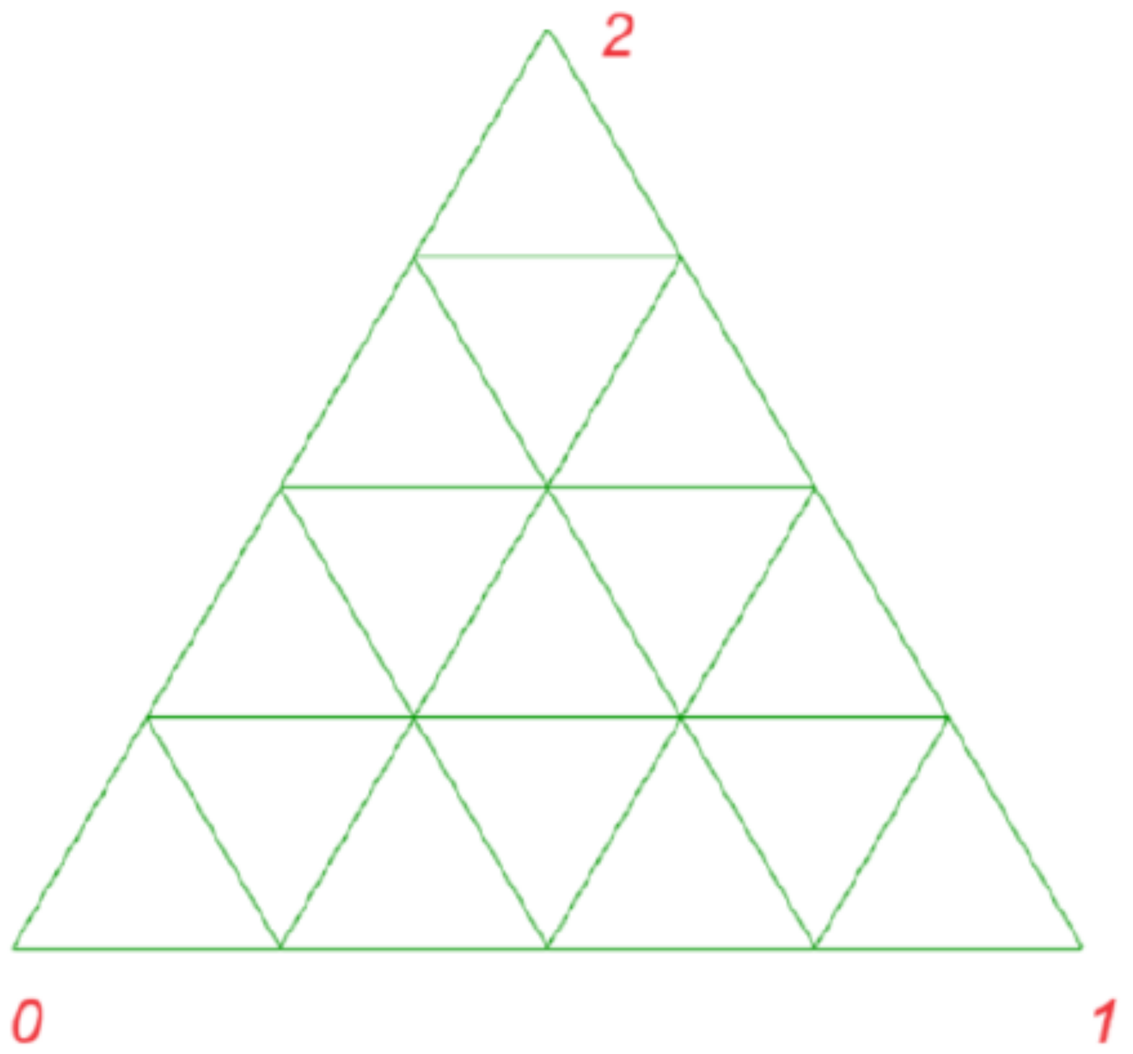
If any of the small triangles has three vertices with three different labels, I will; if never happens, you win....

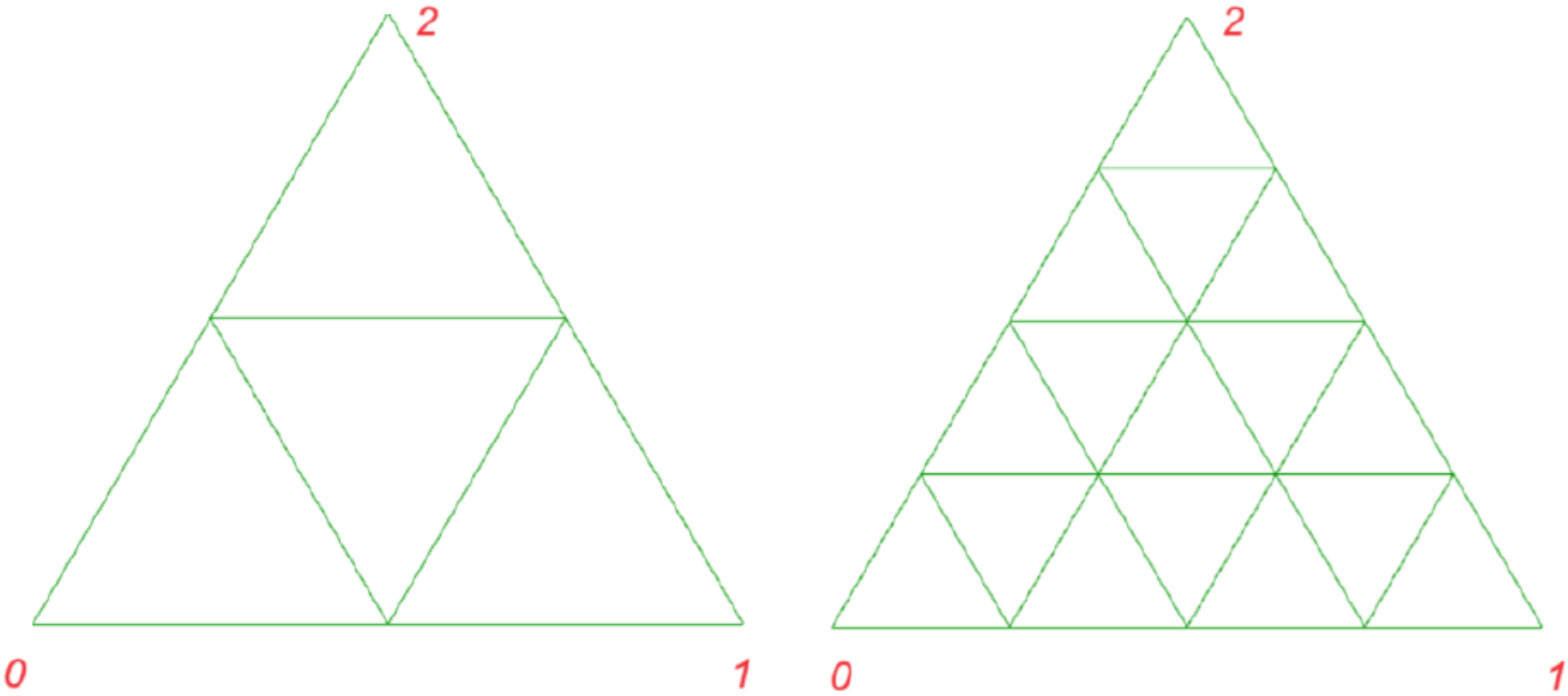
I will give you

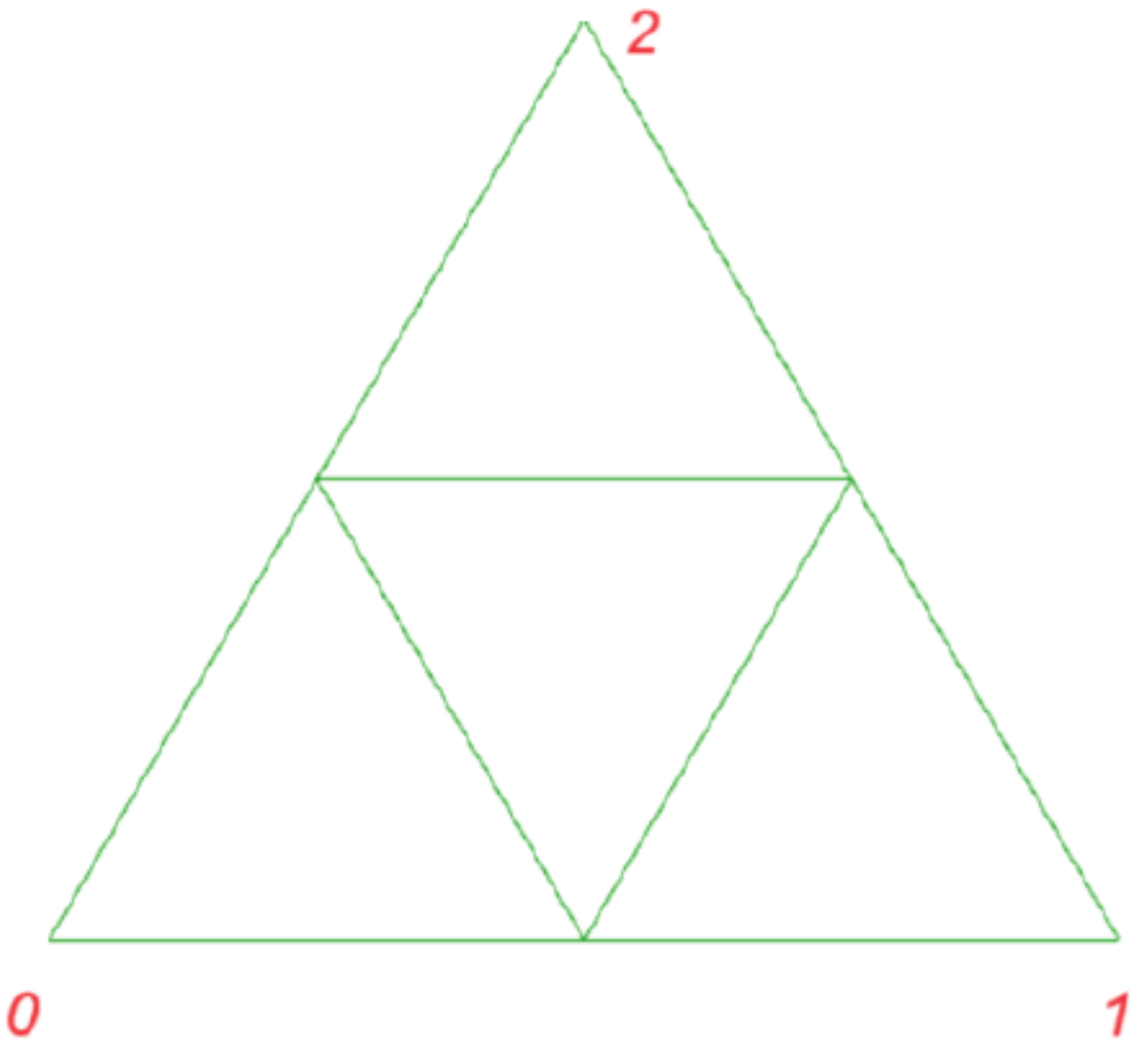
- $100 if you win and we alternate moves,
- $50 if you win and you get two moves for each of mine,
- $25 if you win and you get three moves for each of mine...

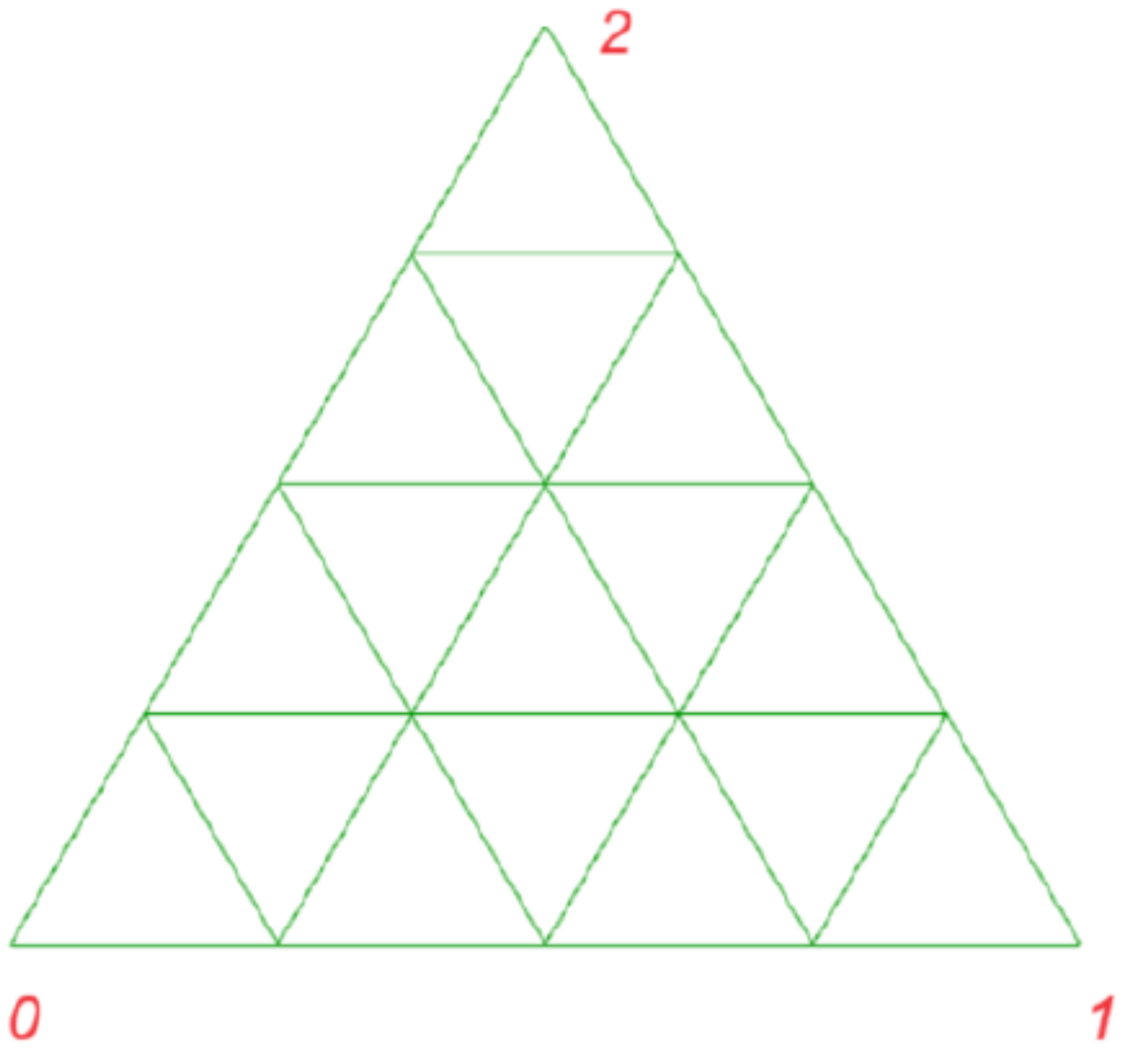
(Can bank moves…)


2
0
0
1
0
1


2
0
1


2
0
1
2
0
1


2
0
1

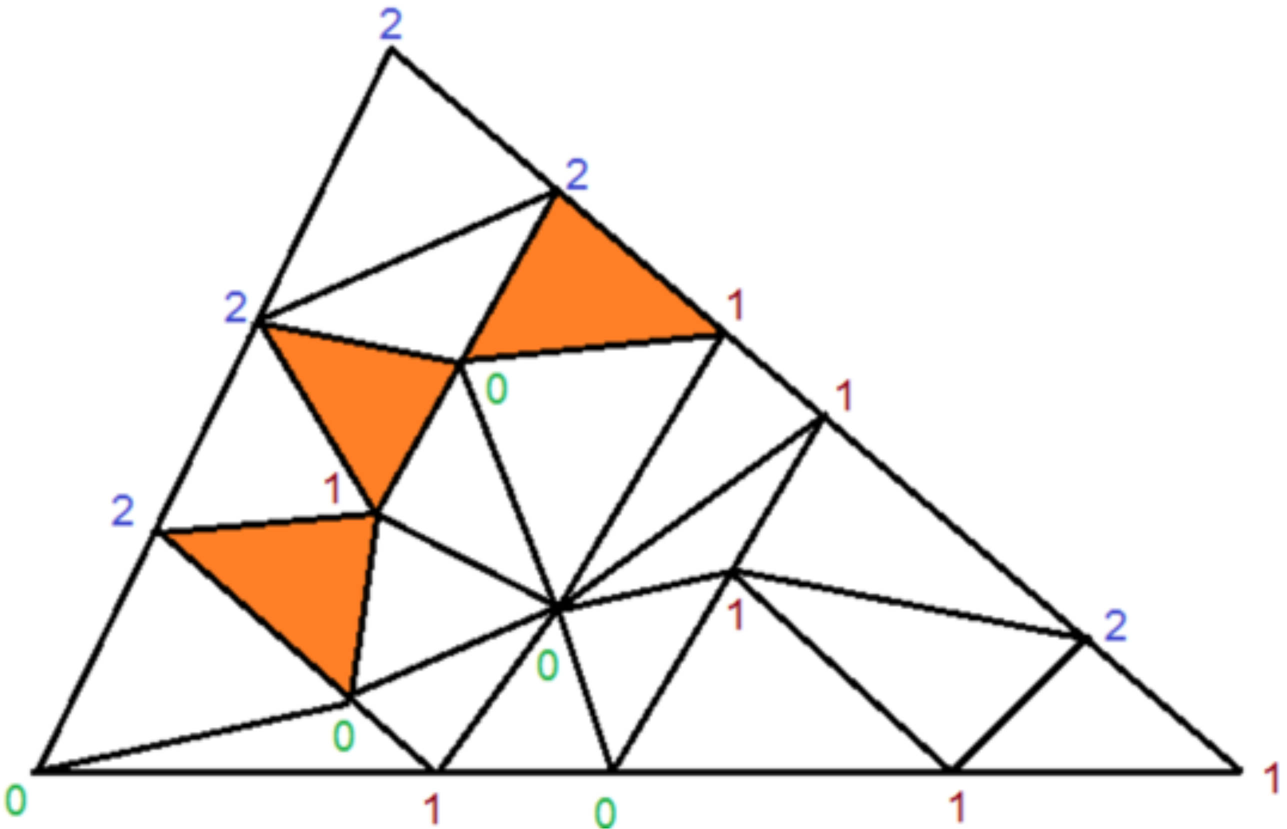

2
0
1

We state the Brouwer Fixed Point Theorem in the case of the standard n-**simplex**:

$$\Delta_n := \{\vec{x} \in \mathbb{R}^{n+1} : x_0 + \cdots + x_n = 1, x_i \geq 0\}.$$

Theorem 16.3.1 (Brouwer Fixed Point Theorem). *Any continuous* $f : \Delta_n \to \Delta_n$ *has a fixed point.*

Lemma 15.1.1 (Sperner's Lemma (2D)). *Consider a triangle whose three vertices are labeled 0, 1 and 2. Sub-divide into non-intersecting triangles, and label the vertices from* $\{0, 1, 2\}$*, subject to each vertex along the three outer edges of the triangle has its label from among those of the edge's endpoints (thus the vertices along the* $1-2$ *edge all have label either* 1 *or* 2*). For the internal vertices, choose any labels among* 0*,* 1*, or* 2*. For every such labeling, there will always be a triangle whose vertices have distinct labels and which has no further division points inside it or along its edges.*



The one-dimensional case is much easier.

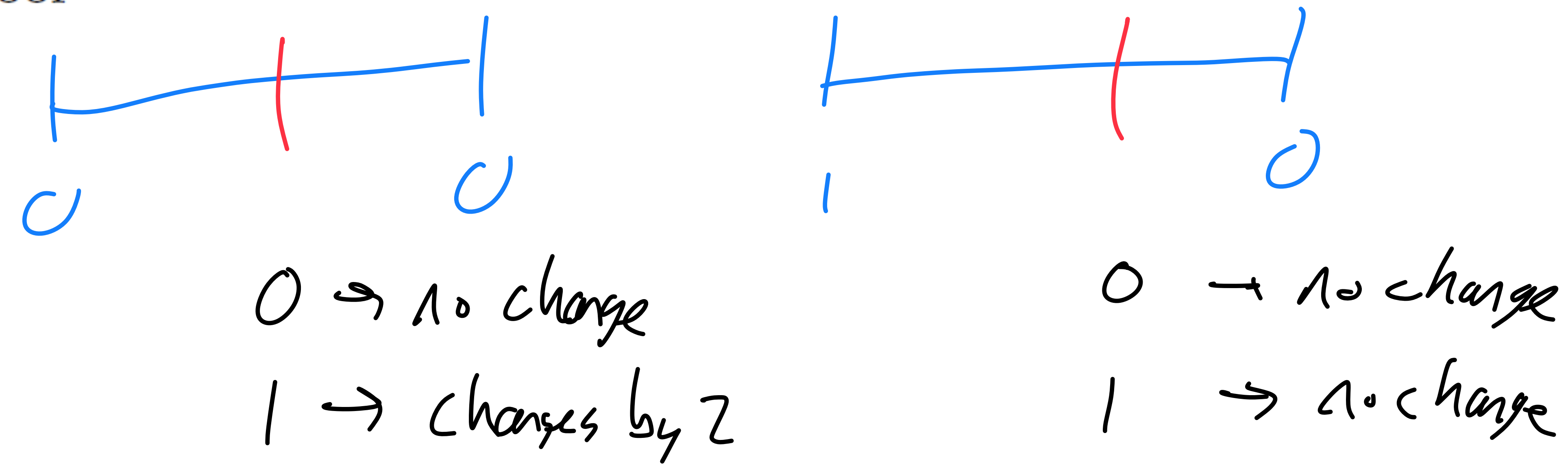
Lemma 15.1.2 (Sperner's Lemma (1D)). *Consider a line segment with left point labeled 0 and right point labeled 1. Sub-divide into non-intersecting segments, with the division points labeled 0 or 1. For every such labeling, there will always be an edge whose endpoints have distinct labels and which has no further division points inside it.*

First proof

The one-dimensional case is much easier.

Lemma 15.1.2 (Sperner's Lemma (1D)). *Consider a line segment with left point labeled 0 and right point labeled 1. Sub-divide into non-intersecting segments, with the division points labeled 0 or 1. For every such labeling, there will always be an edge whose endpoints have distinct labels and which has no further division points inside it.*

Second proof



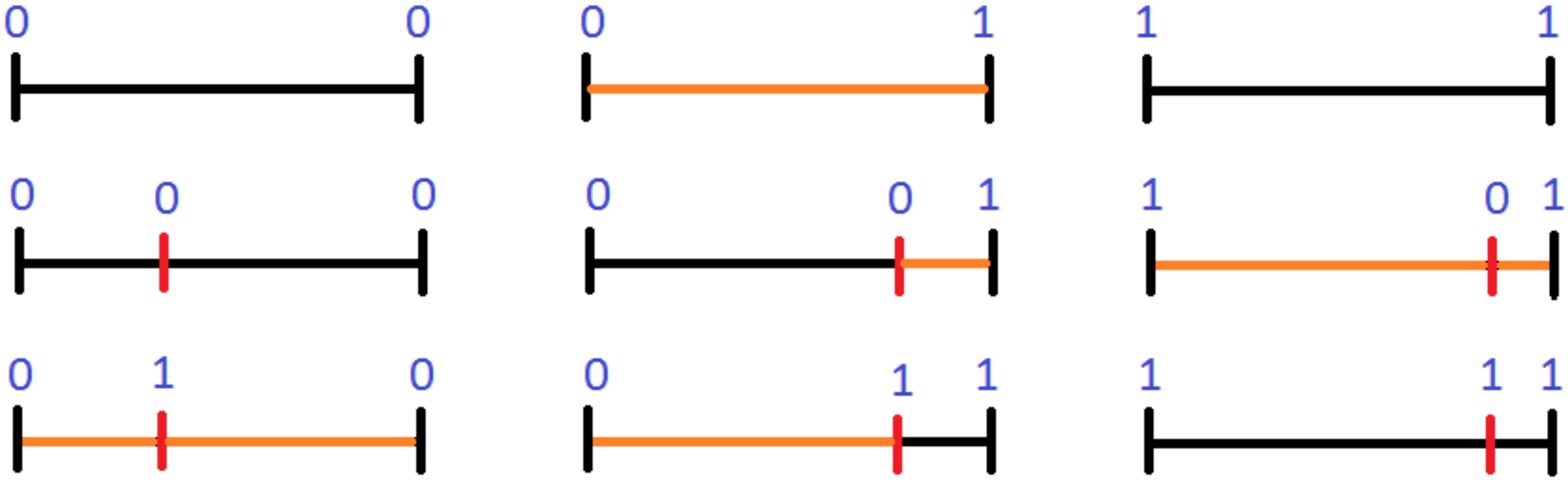


Figure 3. The various cases from the 1-dimensional Sperner proof. Notice in each of the six cases, the change in the number of 0–1 segments is even.

Proof of Sperner's Lemma: Parity!

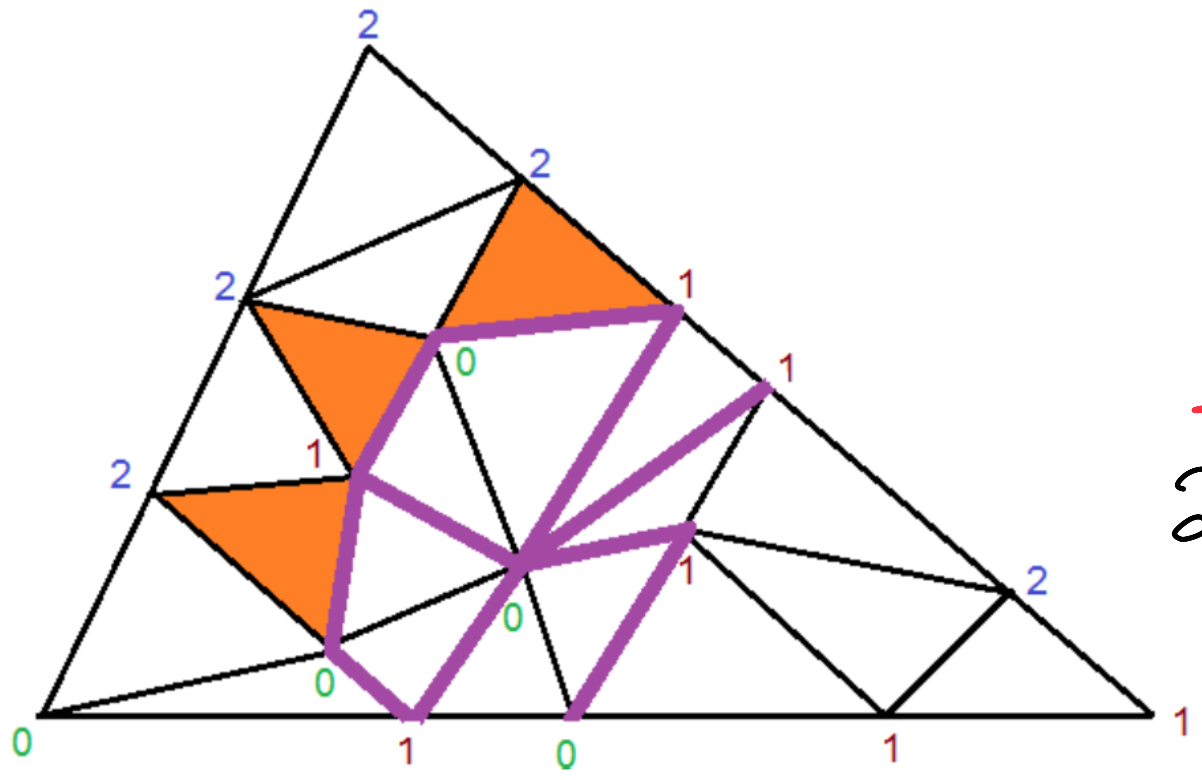


Figure 4. A two-dimensional example of Sperner's Lemma, with the 0–1–2 triangles marked and the internal 0–1 segments highlighted.

odd # of 0-1 segments on bottom

T_{011} → gives two 0-1 segments

T_{001} → gives two 0-1 segments

T_{012} → gives one 0-1 segment

even

$2(\# T_{011}) + 2(\# T_{001}) + 1(\# T_{012})$ ODD

$= \#$ (0-1 segments)

$+ \#$ (internal 0-1 segments)

even

$= 2(\#$ internal 0-1 segments$)$

$+ (\#$ external 0-1 segments$)$

ODD

Improving Sperner's Game:

- Try Squares and 4 colors
- No restrictions on border
- Have at least 3 to win
-

Improving Sperner's Game:

- higher dimensions
- different shapes: torus
- polygon with n-sides
- label edges or faces
- more labels on triangle

- real labels, maximize function

Improving Sperner's Game: Polymath Jr REU '25
Joint with David Gold, Commonwealth School (From Teachers as Scholars program)

- 1D: Subtract 1 for guaranteed segment, whomever has more (00 and 11 vs 01) wins.

- 2D: Win if beat expected number.

- Are there constructive strategies to win?

- Higher dimensions….

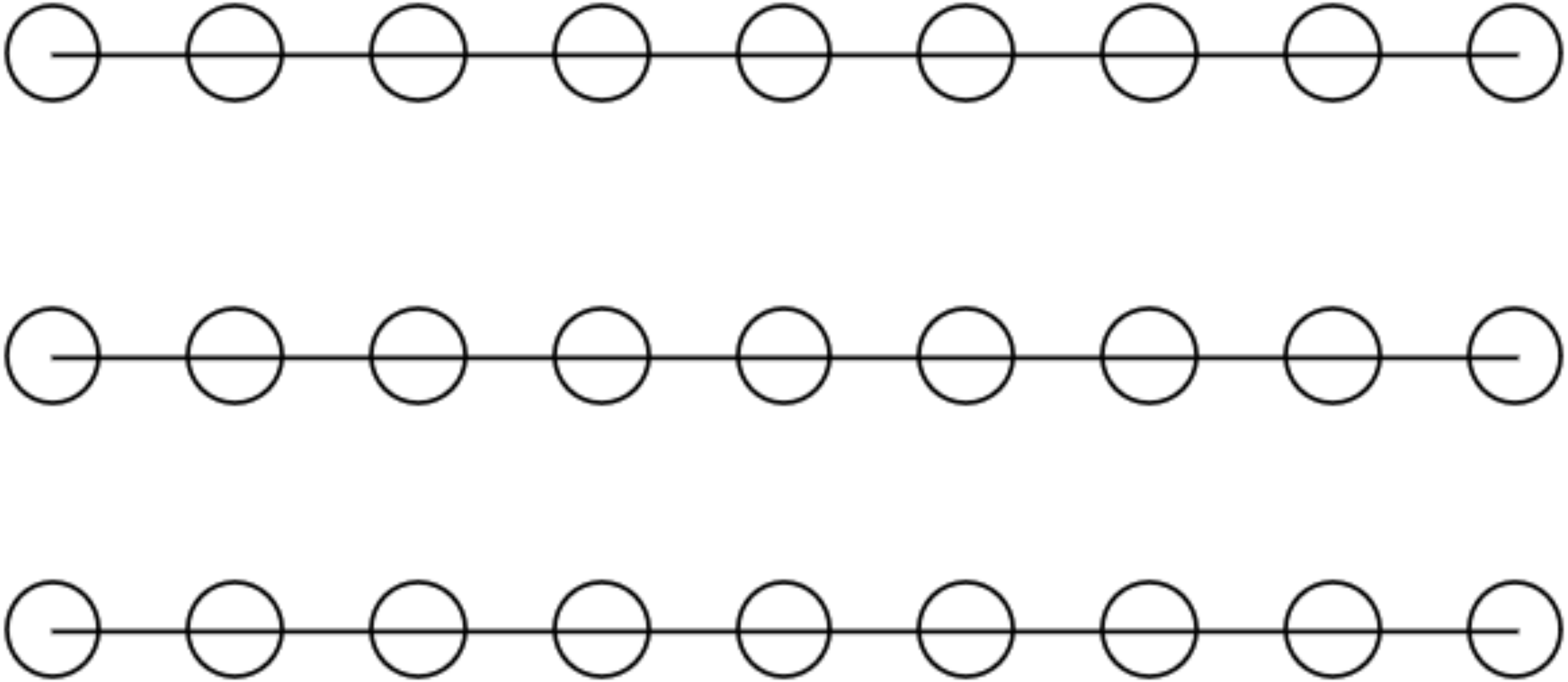
Improving Sperner’s Game:

- Linearity of expectation (mean); variance.

```
interior012prime[numdo_] := Module[{},
  count = 0; labels = {2, 3, 5};
  For[n = 1, n ≤ numdo, n++,
   {
    triangle = Product[labels[[RandomInteger[{1, 3}]]], {k, 1, 3}];
    (* labels 2, 3 or 5; only way get all three is if product 30 *)
    If[triangle == 30, count = count + 1];
   }];
  Print["Observed percentage 012 Interior Triangles: ",
   SetAccuracy[100.0 count / numdo, 6]];
  Print["Predicted percentage: ", SetAccuracy[100.0 * 2 / 9, 6]];
 ]

Timing[interior012prime[10 000 000]]

Observed percentage 012 Interior Triangles: 22.22155

Predicted percentage: 22.22222

{32.4219, Null}
```

Improving Sperner's Game:

- **Simple strategies**



```
Graphics[{Circle[{0, 0}, 1], Circle[{4, 0}, 1], Circle[{8, 0}, 1], Circle[{12,  0}, 1], Circle[{16, 0}, 1], Circle[{20, 0}, 1], Circle[{24, 0}, 1], Circle[{28, 0}, 1], Circle[{32, 0}, 1], Line[{{0, 0}, {32, 0}}]}]
```

THANKS!

If interested in participating as an undergrad researcher or a mentor in Polymath Jr in the future email me at: sjm1@williams.edu

https://preview.redd.it/particle-man-they-might-be-giants-15-they-have-a-fight-v0-71yvgiw11w6c1.png?auto=webp&s=24a0c10ab1ef44ee30ebb766d94862b732a2c8e4

$$f(x) = x$$

$$f(x) = a$$

$$f(x) + x = x + a$$

$$\underbrace{f(x) + x - a}_{g_a(x)} = x$$