How low can we go? Lower order terms in CLTs from Benford's Law to Elliptic Curves

Steven J Miller
Williams College

Steven.J.Miller@williams.edu
http://www.williams.edu/go/math/sjmiller

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Outline

- Review of Central Limit Theorem results.
- Applications to Benford's Law.
- Applications to Elliptic Curves.

Central Limit Theorems

General Statement

Central Limit Theorem

Let $X_1, \dots, X_N$ be nice iiddrv with mean μ and variance σ^2. Then

$$Y_N = \frac{X_1 + \cdots + X_N}{N}, \quad \lim_{N \to \infty} \frac{Y_N - \mathbb{E}[Y_N]}{\mathrm{StDev}(Y_N)} \longrightarrow N(0,1).$$

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- What is nice? Finite higher moments:

$$k^{\text{th}} \text{ centered moment } = \int_{-\infty}^{\infty} (x-\mu)^k f(x) dx.$$

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- What is nice? Finite higher moments:

$$k^{\text{th}} \text{ centered moment } = \int_{-\infty}^{\infty} (x - \mu)^k f(x) dx.$$

- Speed of convergence controlled by higher moments (especially third).

Overview of Benford's Law

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of d base B is $\log_B\left(\frac{d+1}{d}\right)$; base 10 about 30% are 1s.

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- Not all data sets satisfy Benford's Law.
  - ⋄ Long street $[1, L]$: $L = 199$ versus $L = 999$.
  - ⋄ Oscillates between $1/9$ and $5/9$ with first digit 1.
  - ⋄ Many streets of different sizes: close to Benford.

Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L-functions, characteristic polynomials
- iterates of the $3x + 1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Applications

- analyzing round-off errors
- determining the optimal way to store numbers
- detecting tax and image fraud, and data integrity

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Key observation: $\log_{10}(x) = \log_{10}(\widetilde{x})$ mod 1 if and only if x and $\widetilde{x}$ have the same leading digits. Thus often study $y = \log_{10} x$.

Equidistribution and Benford's Law

Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a,b]$ tends to $b-a$:

$$\frac{\#\{n \le N : y_n \bmod 1 \in [a,b]\}}{N} \ \rightarrow \ b-a.$$

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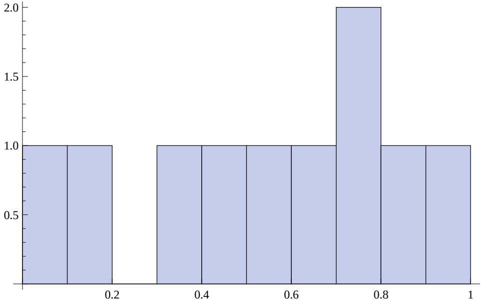
Theorem: Kronecker, Weyl

$\beta \notin \mathbb{Q}$, $n\beta$ is equidistributed mod 1.

Example of Equidistribution: $n\sqrt{\pi}$ mod 1



$n\sqrt{\pi}$ mod 1 for $n \leq 10$

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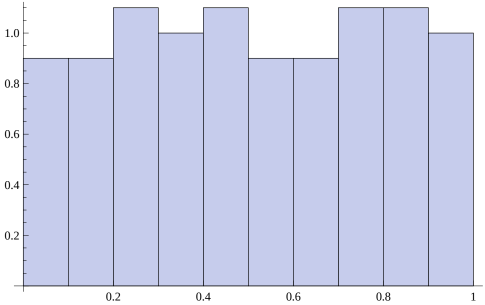

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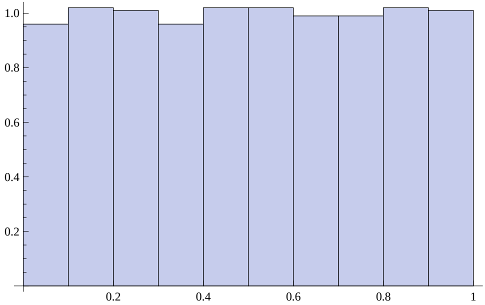

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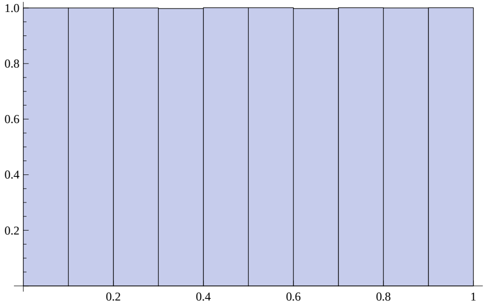

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Logarithms and Benford's Law

Fundamental Equivalence

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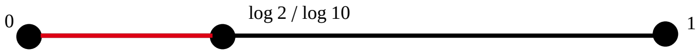

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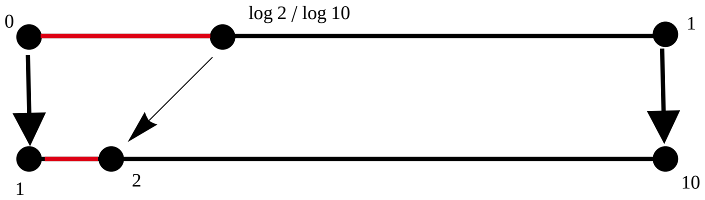

Benford's Law and the $3x + 1$ Problem

$3x+1$ Problem

- Kakutani (conspiracy), Erdös (not ready).
- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k || 3x+1$.

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  2-path $(1,1)$, 5-path $(1,1,2,3,4)$.
  m-path: $(k_1, \dots, k_m)$.

Structure Theorem: Sinai, Kontorovich-Sinai

Theorem (Sinai, Kontorovich-Sinai)

k_i-values are i.i.d.r.v. (geometric, $1/2$):

$$\mathbb{P}\left(\frac{\log_2\left[\frac{x_m}{\left(\frac{3}{4}\right)^m x_0}\right]}{\sqrt{2m}} \leq a\right) = \mathbb{P}\left(\frac{S_m - 2m}{\sqrt{2m}} \leq a\right)$$

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$3x + 1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \to \infty$, $x_m/(3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first N iterates of the $3x + 1$ map are within $2N^{-1/36}$ of the Benford probabilities.

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- Failed Proof: lattices, bad errors.
- CLT: $(S_m - 2m)/\sqrt{2m} \to N(0,1)$.
- Quantified Equidistribution: $I_\ell = \{\ell M, \dots, (\ell+1)M - 1\}$, $M \ll m^{1/2}$
  $\log_B 2$ of irrationality type $\kappa < \infty$:

  $$\#\{k \in I_\ell : k \log_B 2 \bmod 1 \in [a,b]\} = M(b-a) + O(M^r),$$

  $r = 1 + \epsilon - 1/\kappa < 1$.

Irrationality Type

Irrationality type

α has irrationality type κ if κ is the supremum of all γ with

$$\underline{\lim}_{q\to\infty} q^{\gamma+1} \min_p \left|\alpha - \frac{p}{q}\right| = 0.$$

- Algebraic irrationals: type 1 (Roth's Thm).
- Theory of Linear Forms: $\log_B 2$ of finite type.

Linear Forms

Theorem (Baker)

$\alpha_1, \dots, \alpha_n$ algebraic numbers height $A_j \geq 4$, $\beta_1, \dots, \beta_n \in \mathbb{Q}$ with height at most $B \geq 4$,

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n.$$

If $\Lambda \neq 0$ then $|\Lambda| > B^{-C\Omega \log \Omega'}$, with $d = [\mathbb{Q}(\alpha_i, \beta_j) : \mathbb{Q}]$, $C = (16nd)^{200n}$, $\Omega = \prod_j \log A_j$, $\Omega' = \Omega / \log A_n$.

Gives $\log_{10} 2$ of finite type, with $\kappa < 1.2 \cdot 10^{602}$:

$$\left|\log_{10} 2 - p/q\right| = \left|q \log 2 - p \log 10\right| / q \log 10.$$

$3x+1$ Data: random 10,000 digit number, $2^k || 3x+1$

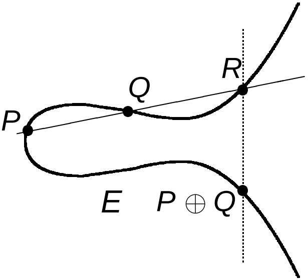
80,514 iterations ($(4/3)^n = a_0$ predicts 80,319);
$\chi^2 = 13.5$ (5% 15.5).

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

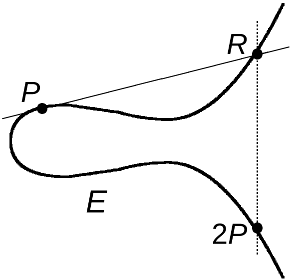
Elliptic Curves

Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.



Addition of distinct points P and Q



Adding a point P to itself

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$$

Riemann Zeta Function

$$\zeta(s) \ = \ \sum_{n=1}^{\infty} \frac{1}{n^s} \ = \ \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) \ = \ \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) \ = \ \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

General L-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) = \Lambda_\infty(s,f)L(s,f) = \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (GRH):

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Elliptic curve L-function

$E : y^2 = x^3 + ax + b$, associate L-function

$$L(s,E) \;=\; \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x,y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

Birch and Swinnerton-Dyer Conjecture

Rank of group of rational solutions equals order of vanishing of $L(s,E)$ at $s = 1/2$.

1-Level Density

L-function $L(s,f)$: by RH non-trivial zeros $\frac{1}{2}+i\gamma_{f,j}$.
N_f: analytic conductor.
$\varphi(x)$: compactly supported even Schwartz function.

$$D_{1,f}(\varphi) \;=\; \sum_j \varphi\left(\frac{\log N_f}{2\pi}\gamma_{f,j}\right)$$

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Katz-Sarnak Conjecture:

$$D_{1,\mathcal{F}}(\varphi) = \lim_{N\to\infty}\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{1,f}(\varphi) = \int \varphi(x)\rho_{G(\mathcal{F})}(x)dx$$
$$= \int \widehat{\varphi}(u)\widehat{\rho}_{G(\mathcal{F})}(u)du.$$

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} \quad = \quad -\frac{d}{ds}\log\zeta(s) \;=\; -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

Explicit Formula (Contour Integration)

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1} \\ &= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s). \end{aligned}$$

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Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.$$

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Contour Integration (see Fourier Transform arising):

$$\int -\frac{\zeta'(s)}{\zeta(s)}\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) e^{-\sigma\log p} e^{-it\log p} ds.$$

Interplay b/w zeros and coefficients

1-Level Expansion

$$D_{1,\mathcal{F}}(\phi) = \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{j}\phi\left(\frac{\log N_E}{2\pi}\gamma_E^{(j)}\right)$$

$$\begin{aligned} = & -\frac{2}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{p}\frac{\log p}{\log N_E}\frac{1}{p}\widehat{\phi}\left(\frac{\log p}{\log N_E}\right)a_E(p) \\ & -\frac{2}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{p}\frac{\log p}{\log N_E}\frac{1}{p^2}\widehat{\phi}\left(2\frac{\log p}{\log N_E}\right)a_E^2(p) \\ & +\widehat{\phi}(0)+\phi(0)+O\left(\frac{\log\log N_E}{\log N_E}\right)\end{aligned}$$

Inputs

One-parameter family:

$$\mathcal{E} : y^2 = x^3 + A(T)x + B(T), \quad A(T), B(T) \in \mathbb{Z}[T],$$

take $t \in \mathbb{Z}$.

Let

$$A_{r,\mathcal{F}}(p) \;=\; \sum_{t(p)} a_t^r(p), \quad r = 1 \text{ or } 2.$$

For many families

$$\begin{aligned}(1) : A_{1,\mathcal{F}}(p) &= -rp \;+\; O(1)\\ (2) : A_{2,\mathcal{F}}(p) &= p^2 \;+\; O(p^{3/2})\end{aligned}$$

Main Term

Theorem: M– '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

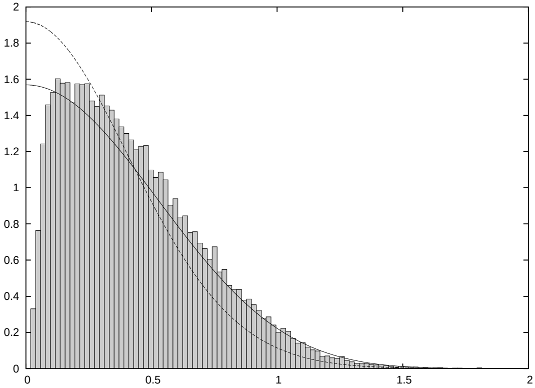
$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t\in\mathcal{F}_N} \sum_j \varphi\left(\frac{\log N_{E_t}}{2\pi}\gamma_{E_t,j}\right) = \int \varphi(x)\rho_{\mathcal{G}}(x)dx + r\varphi(0)$$

where

$$\mathcal{G} = \begin{cases} SO & \text{if half odd} \\ SO(\text{even}) & \text{if all even} \\ SO(\text{odd}) & \text{if all odd} \end{cases}$$

Confirm Katz-Sarnak for main term

Modeling lowest zero (data & calculations from Duc Khiem Huynh)



Lowest zero $L_{E_{11}}(s, \chi_d)$, $0 < d < 400,000$ (bar chart), lowest eigenvalue SO(2N) w' $N_{\rm eff}$ (solid), standard N_0 (dashed).

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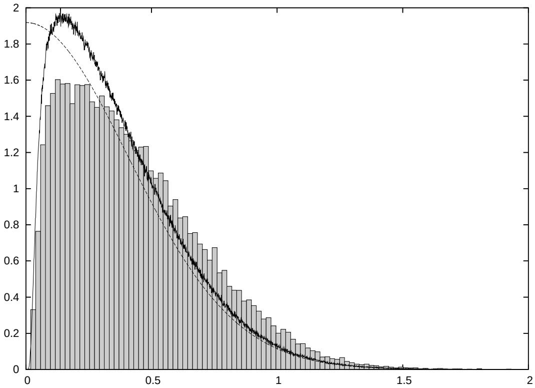

Lowest zero $L_{E_{11}}(s, \chi_d)$, $0 < d < 400,000$ (bar chart), lowest eigenvalue SO(2N) w' $N_0 = 12$ (solid) w' discretisation and w' standard $N_0 = 12.26$ (dashed) w/o discretisation.

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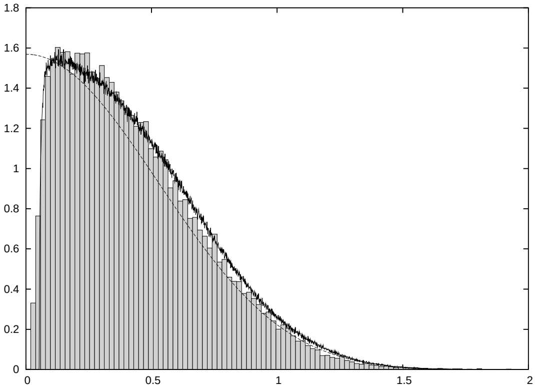

Lowest zero $L_{E_{11}}(s, \chi_d)$, $0 < d < 400,000$ (bar chart), lowest eigenvalue SO(2N) w' N_{eff} = 2 (solid) w' discretisation and w' $N_{\text{eff}} = 2.32$ (dashed) w/o discretisation.

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