$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots = \sum_{k=0}^{\infty} r^k$$

Introduction to Sums

Steven J Miller, Williams College

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$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

These math outreach lectures are supported in part by the Journal of Number Theory and the Teachers as Scholars program; it is a pleasure to thank them for their support.

Goals

Want to learn how to evaluate sums.

Will see a variety of techniques, including Induction.

Part I: Induction

JOURNAL OF NUMBER THEORY

Teachers As Scholars
henry.bolter@gmail.com

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Notation

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Imagine we have a sequence a_1, a_2, a_3, a_4, ….

Perhaps $a_n = n^2$ so the sequence is 1, 4, 9, 16, …..
Or maybe a_n is the n^{th} prime, so the sequence is 2, 3, 5, 7, …..

By $\sum_{n=4}^{9} a_n$ we mean $a_4 + a_5 + a_6 + a_7 + a_8 + a_9$. This is concise notation, and saves us the trouble of writing everything each time.

Sometimes convenient to start with a_0.

Induction

One of the most important techniques we have for proving results.

Say we have some statement P(n). Perhaps P(n) is “the sum of the first n integers is n(n+1)/2”.

We can check this for various n; every time we check it is true but that is NOT the same as a proof!

Example: $\frac{16}{64} = \frac{1}{4}$, $\frac{19}{95} = \frac{1}{5}$, $\frac{49}{98} = \frac{1}{2}$ but $\frac{12}{24}$ is not $\frac{1}{4}$.

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Say we have some statement P(n). Perhaps P(n) is “the sum of the first n integers is n(n+1)/2”.

Imagine we can show the following two statements are true.

1. P(1) is true, and
2. Whenever P(n) is true then P(n+1) is true.

If we can do this we now have P(n) is true for all n!

(Note: Sometimes we start at n=0 not n=1)

Induction

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1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Why does this imply that it holds for all n?

Induction (Box, Dirichlet, Pigeonhole Principle)

Imagine we can show the following two statements are true.

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Take n=1: thus the second becomes P(1) true implies P(2) true

P(1) is true

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THEREFORE since P(1) is true we now know P(2) is true.

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We know P(1) and P(2) are true.

Take n=2: thus the second becomes P(2) true implies P(3) true

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Imagine we can show the following two statements are true.

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

We know P(1), P(2) and P(3) are true.

Take n=3: thus the second becomes P(3) true implies P(4) true

P(3) is true

P(3) true implies P(4) true

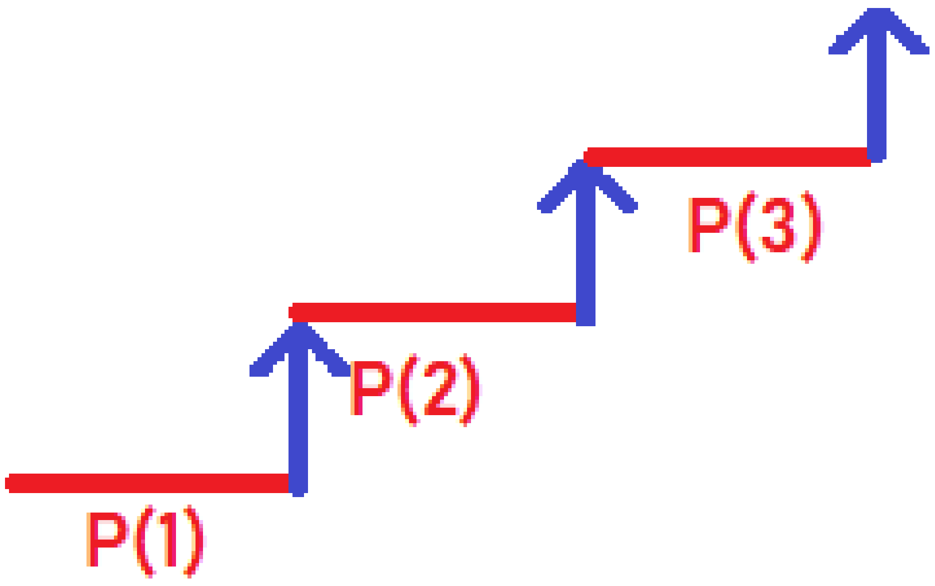
THEREFORE since P(3) is true we now know P(4) is true. AND SO ON!

Induction (Box, Dirichlet, Pigeonhole Principle)

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

This is often viewed
as a staircase.



Example: P(n): 1 + 2 + … + n = n(n+1)/2

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1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

We will prove this by induction. There are two steps.

First we prove P(1) is true, then we show IF P(n) is true THEN P(n+1) is true.

Example: $P(n)$: $1 + 2 + \ldots + n = n(n+1)/2$

To prove $P(n)$ is true for all n, must show

1. Base case: $P(1)$ is true, and
2. Inductive Step: Whenever $P(n)$ is true then $P(n+1)$ is true.

Step 1: Base Case: We must show $P(1)$ is true.

Thus we must show that when $n=1$, we have $1 = 1(1+1)/2$.

This however follows immediately!

We are done with the base case.

Example: P(n): $1 + 2 + \ldots + n = n(n+1)/2$

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We are done with the base case. We could try to do n=2 or n=3 to build up intuition, but it is not necessary.

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Extra work: If n=2 let's check: Does 1+2 = 2(2+1)/2? YES!

Extra work: if n=3 let's check: Does 1+2+3 = 3(3+1)/2? YES!

These extra checks are not a substitute for a proof, but the more values of n that work, the more confident we are that it is true.

Example: P(n): $1 + 2 + \ldots + n = n(n+1)/2$

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OK, we now get to assume P(n) is true, we want to prove P(n+1) is true.

What does this mean?

P(n) true means we assume $1 + 2 + \ldots + n = n(n+1)/2$.

We want to prove that P(n+1): $1 + 2 + \ldots + n + (n+1) = (n+1)(n+1+1)/2$ is true.

How should we proceed? When we look at P(n+1), do we see anything related to P(n)?

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What are we assuming we know about 1 + 2 + … + n? We are assuming it equals n(n+1)/2.

Thus let's substitute for 1 + 2 + … + n in 1 + 2 + … + n + (n+1).

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Using the inductive assumption, we have

$1 + 2 + \ldots + n + (n+1) = (1 + 2 + \ldots + n) + (n+1) = n(n+1)/2 + (n+1)$.

Now we just need to show the far right equals our claim, $(n+1)(n+1+1)/2$. How do we add two fractions?

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But $\frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$, which is what we needed to show, completing the proof (as n+2 = n+1+1)!

Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

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We don't need to, but we can check other values of n.

If n=2 does $1^2 + 2^2 = 2(2+1)(2*2+1)/6$? YES!

If n=3 does $1^2 + 2^2 + 3^2 = 3(3+1)(2*3 + 1)/6$? YES!

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Since we are assuming P(n) is true, what do we know?

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P(n+1) is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1)(n+1+1)(2(n+1)+1)/6$, note the right hand side is $(n+1)(n+2)(2n+3)/6$.

What is in common with P(n) and P(n+1)?

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What is in common with P(n) and P(n+1)? We can now substitute….

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So is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (1^2 + 2^2 + \ldots + n^2) + (n+1)^2 = n(n+1)(2n+1)/6 + (n+1)^2$.

We have to combine the fractions – how do we do that?

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We have $\frac{n(n+1)(2n+1)}{6} = \frac{6(n+1)^2}{6} =$??? What is in common with the two fractions? Both have a ....

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We have $\frac{n(n+1)(2n+1)}{6} = \frac{6(n+1)^2}{6} = \frac{(n+1)(n(2n+1)+\ 6(n+1))}{6} = \frac{(n+1)(2n^2+n+6n+6)}{6} = \frac{(n+1)(2n^2+7n+6)}{6}$

Doing some algebra, we see $2n^2 + 7n + 6$ equals (n+2)(2n+3) by FOIL, completing the proof.

Example: P(n): $1 + 3 + \ldots + (2n-1) = n^2$

To prove P(n) is true for all n, must show

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If n=2 does $1 + 3 = 2^2$? YES!

If n=3 does $1 + 3 + 5 = 3^2$? YES!

Rest of the proof is similar to what we've done before....

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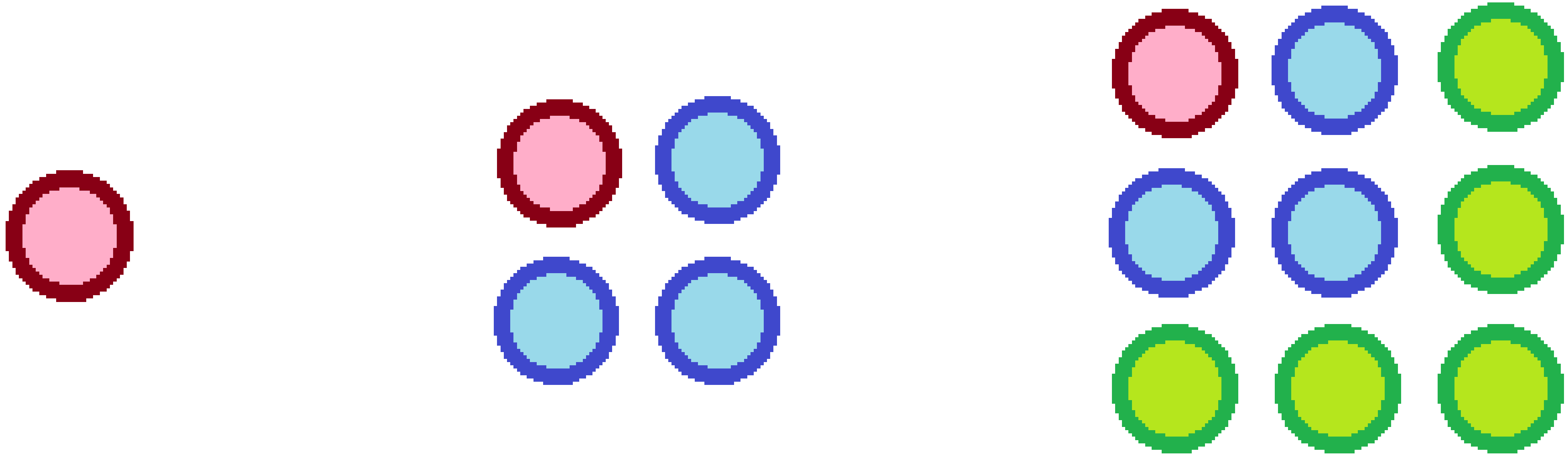
Can prove in other ways than Induction….



Example: P(n): 133 divides $11^{n+1} + 12^{2n-1}$

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Let's try to show P(1) is true: does 133 divide $11^{1+1} + 12^{2*1-1}$?

Example: P(n): 133 divides $11^{n+1} + 12^{2n-1}$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Let's try to show P(1) is true: does 133 divide $11^{1+1} + 12^{2*1-1}$?

Yes, as $11^{1+1} + 12^{2*1-1} = 11^2 + 12 = 121 + 12 = 133$, which is clearly a multiple of 133.

Example: P(n): 133 divides $11^{n+1} + 12^{2n-1}$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Now assume P(n) is true, we must show P(n+1) is true.

Can assume 133 divides $11^{n+1} + 12^{2n-1}$, must show 133 divides $11^{n+1} + 12^{2n-1}$.

$$
\begin{aligned}
11^{(n+1)+1} + 12^{2(n+1)-1} &= 11^{n+1+1} + 12^{2n-1+2} \\
&= 11 \cdot 11^{n+1} + 12^2 \cdot 12^{2n-1} \\
&= 11 \cdot 11^{n+1} + (133 + 11) 12^{2n-1} \\
&= 11 \left(11^{n+1} + 12^{2n-1} \right) + 133 \cdot 12^{2n-1}. \quad \text{(A.6)}
\end{aligned}
$$

By the inductive assumption 133 divides $11^{n+1} + 12^{2n-1}$; therefore, 133 divides $11^{(n+1)+1} + 12^{2(n+1)-1}$, completing the proof.

Getting a feel for the answer....

We showed $1 + 2 + \ldots + n = n(n+1)/2 = n^2/2 + n/2$.

Is this reasonable?

How can we try to get an UPPER BOUND and a LOWER BOUND for the sum?

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Every term in the sum is at most ???
Every term in the sum is at least ???
The number of terms is ???
Thus an upper bound is ???
Thus a lower bound is ???

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How can we try to get an UPPER BOUND and a LOWER BOUND for the sum?

Every term in the sum is at most n

Every term in the sum is at least 1

The number of terms is n

Thus an upper bound is $n*n = n^2$

Thus a lower bound is $1*n = n$.

Note there is a large difference between the upper and lower bounds, need to do better.

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How can we try to get an UPPER BOUND and a LOWER BOUND for the sum?

Every term in the sum is at most n
The number of terms is n
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The last half of the terms are each at least n/2 and there are n/2.
Thus a lower bound is $n/2 * n/2 = n^2/4$.

Now $n^2 / 4 \leq 1 + 2 + \ldots + n \leq n^2$, note these bounds are of the same power in n!

Final thoughts on sums of powers….

Hardest part of the induction is knowing what to PROVE.

How can we find the formula?

Looking at the cases we've done it looks like it is always a polynomial of degree one higher than the power, constant term is zero, leading term (if sum of k^{th} powers) is $n^{k+1} / (k+1)$.

Note 2 points determine a line, 3 points a quadratic (parabola), 4 a cubic, and so on; we can evaluate the sum for a few points and then INTERPOLATE and figure out the polynomial!

Homework: Prove $1^3 + 2^3 + \ldots + n^3 = n^2 (n+1)^2 / 4$.

False proofs by Induction

The following is my favorite false proof by induction. Where is the mistake?

P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Let's try to prove this by induction. We must show:

1. Base Case: In any group with 1 person, everyone has the same name.
2. Inductive Step: If everyone in a group of size n has the same name, then everyone in a group of size n+1 has the same name.

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PROOF OF BASE CASE: This follows immediately, as....

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P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Let's try to prove this by induction. We must show:

1. Base Case: In any group with 1 person, everyone has the same name.

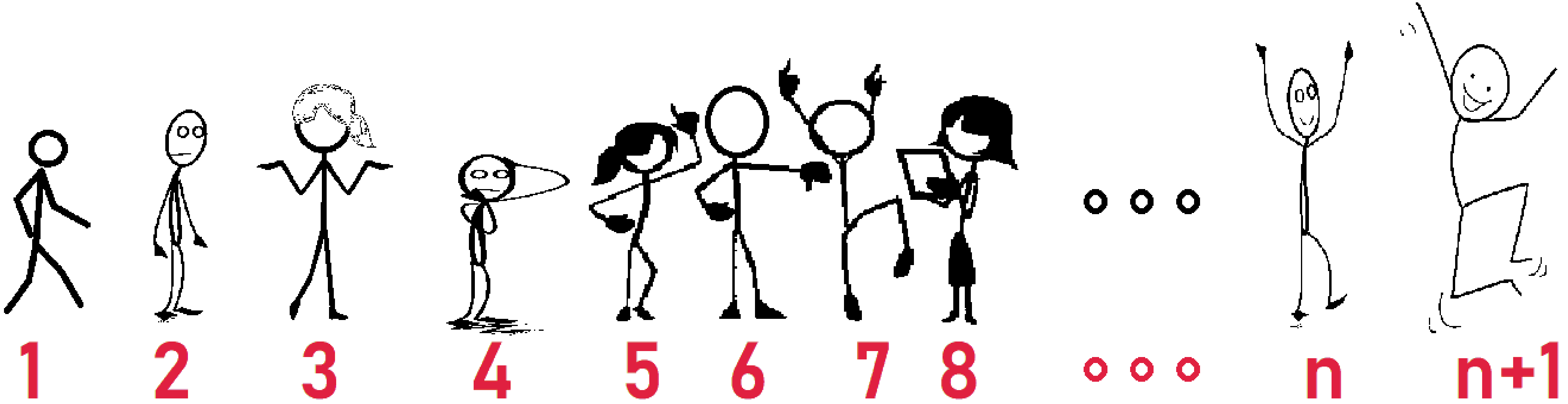
PROOF OF BASE CASE: This follows immediately, as there is only one person in the group, so clearly everyone in the group has the same name!

False proofs by Induction

P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Inductive Step: If everyone in a group of size n has the same name, then everyone in a group of size n+1 has the same name.

“PROOF” OF INDUCTIVE STEP: We assume everyone in a group of size n has the same name, must show true for a group of size n+1. Consider a group of n+1 people. How can we use the inductive assumption (all groups of size n have all with the same name)? Can you find some groups of size n?



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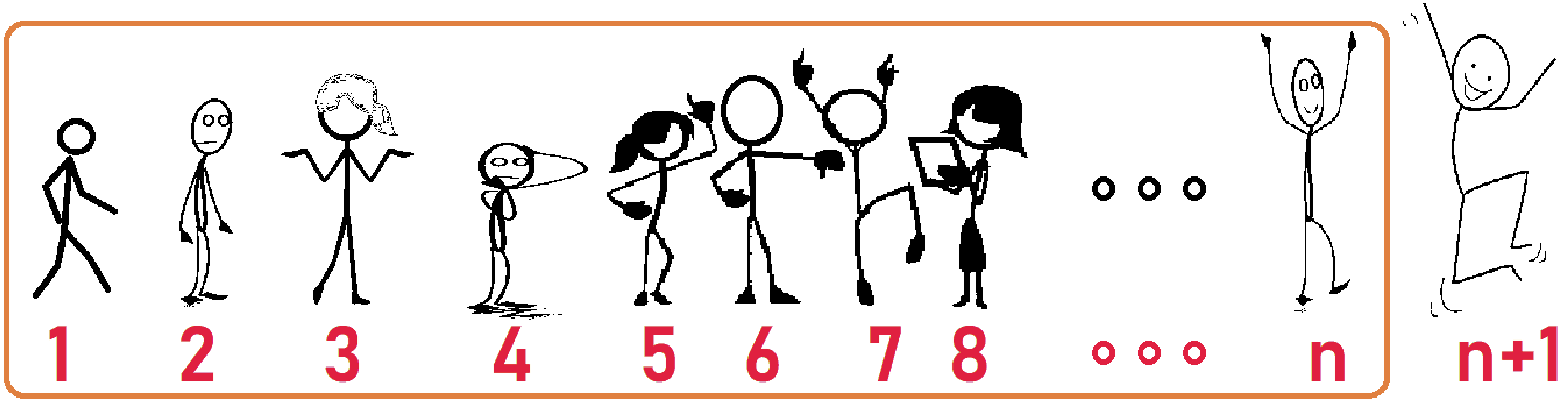

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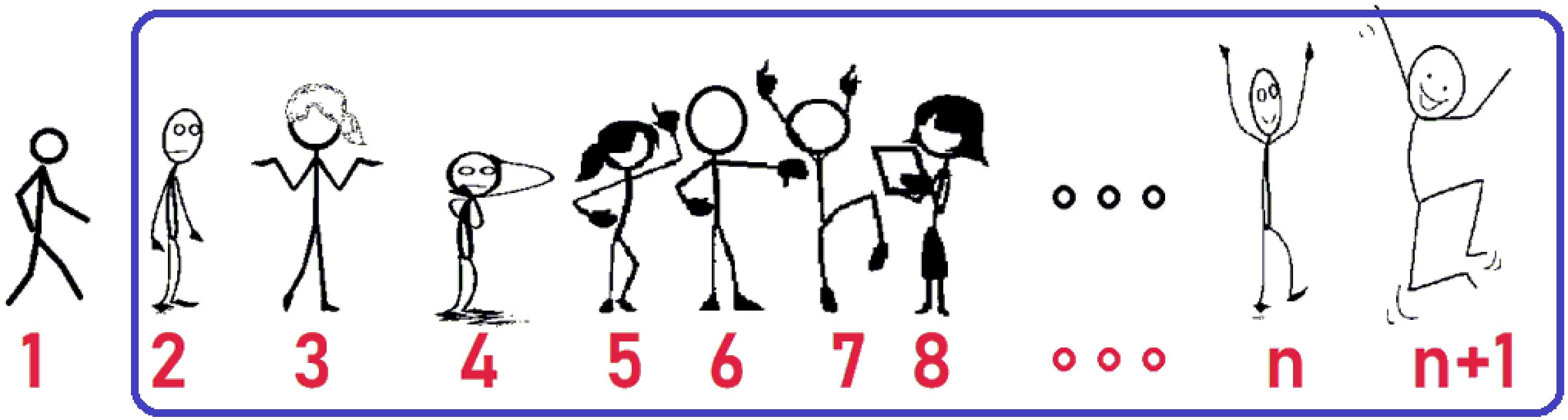

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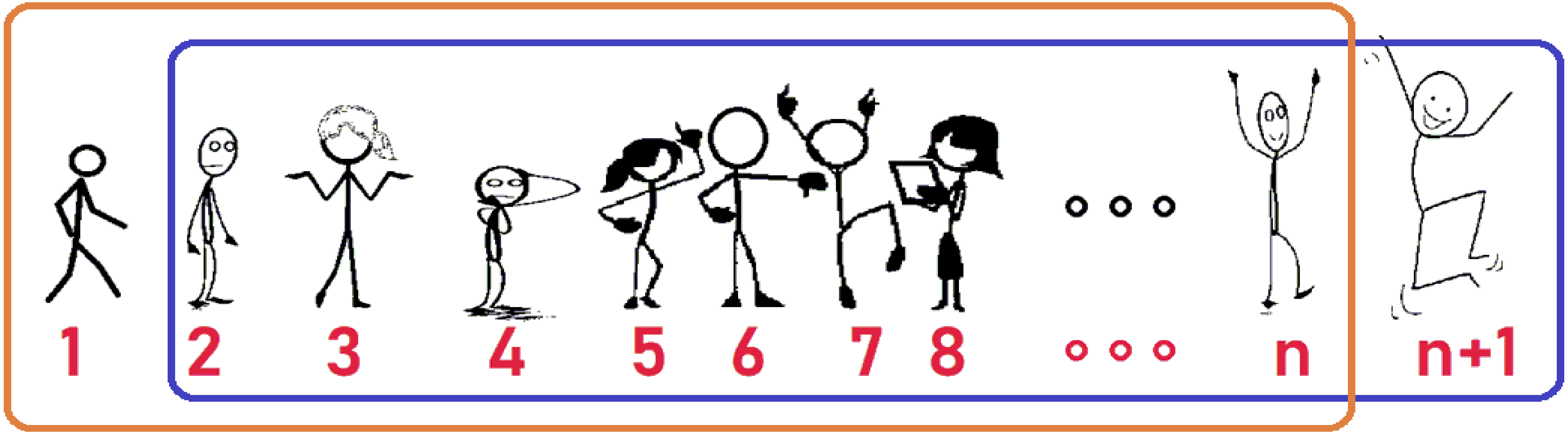

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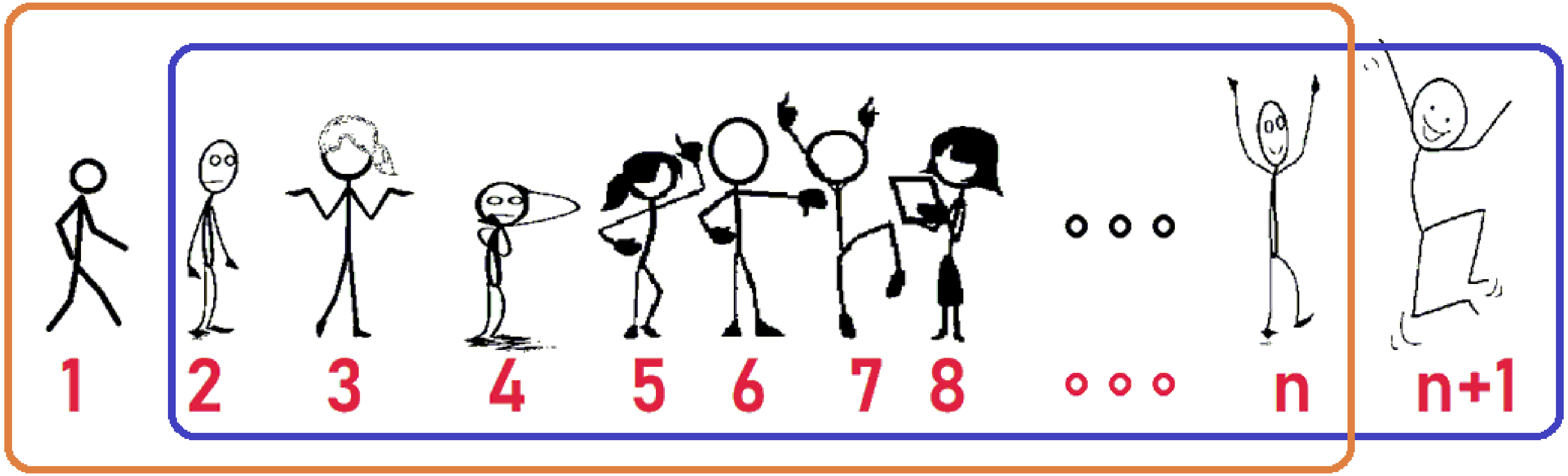
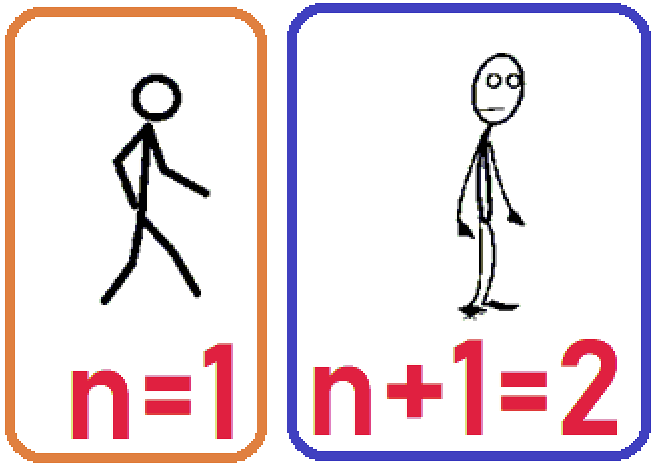


False proofs by Induction

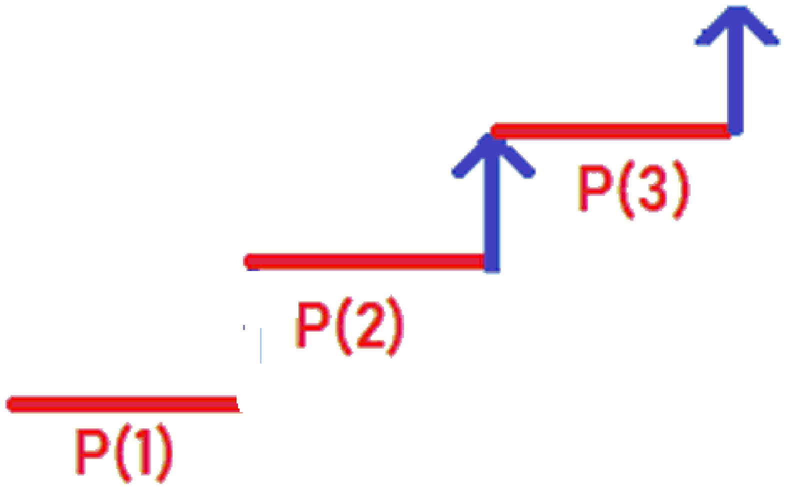
P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Inductive Step: If everyone in a group of size n has the same name, then everyone in a group of size n+1 has the same name.

"PROOF" OF INDUCTIVE STEP: The mistake is we drew this for a "large" n. Remember we must show for ANY n that if P(n) is true then P(n+1) is true. If n is 2 or more then there is a person in both groups, but if n=1 there is not!



When n=1 we see there is no overlap!



Part II: The Geometric Series Formula

JOURNAL OF NUMBER THEORY

Teachers As Scholars
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From Shooting Hoops
to the Geometric Series Formula

The Geometric Series Formula

The Geometric Series Formula is one of the most important in mathematics. It is one of the few sums we can evaluate exactly.

If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots = \frac{1}{1-r}$.

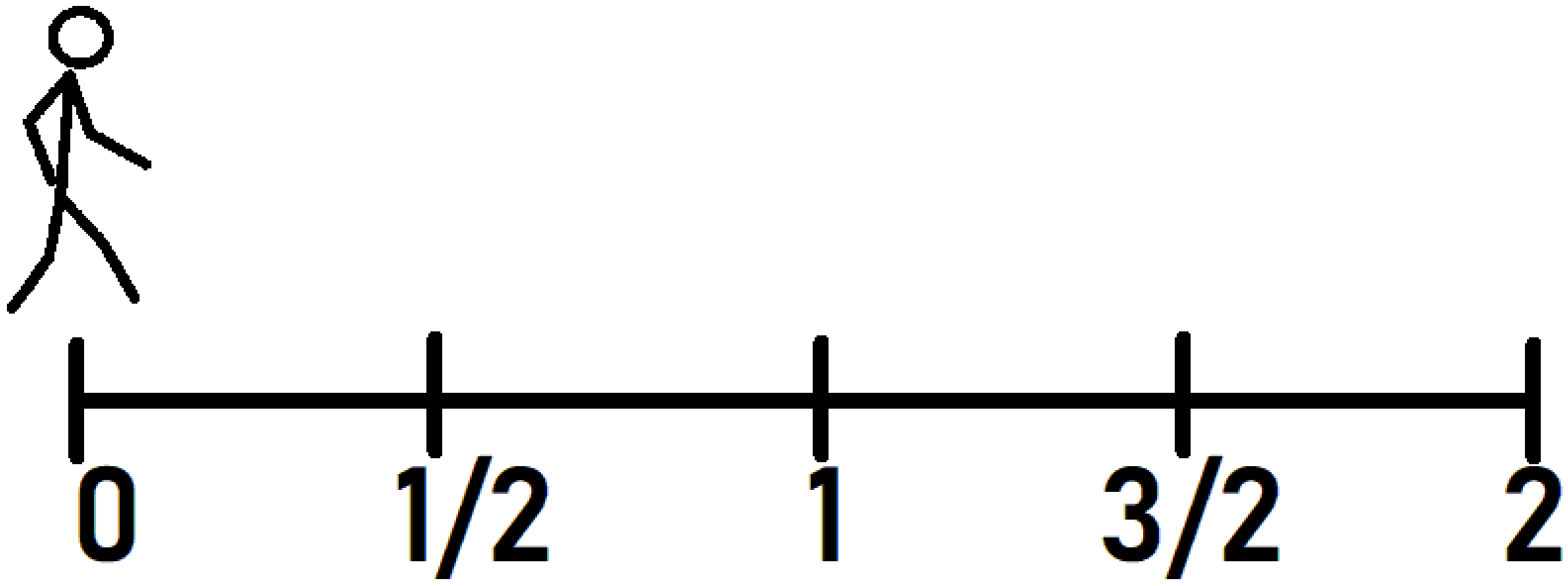
This is often proved by first computing the finite sum, up to r^n, and taking a limit. Note since $|r| < 1$ that each term r^n gets small fast.....

The Geometric Series Converges if |r| < 1

$$1 + r + r^2 + r^3 + r^4 + \cdots = \frac{1}{1-r}.$$

Why does this converge? Take r = ½. We then have 1 + ½ + ¼ + …. = $\frac{1}{1-\frac{1}{2}} = 2$,

and we can view this as we start at 0, and each step covers half the distance to 2. We thus never reach it in finitely many steps, but we cover half the ground each time.



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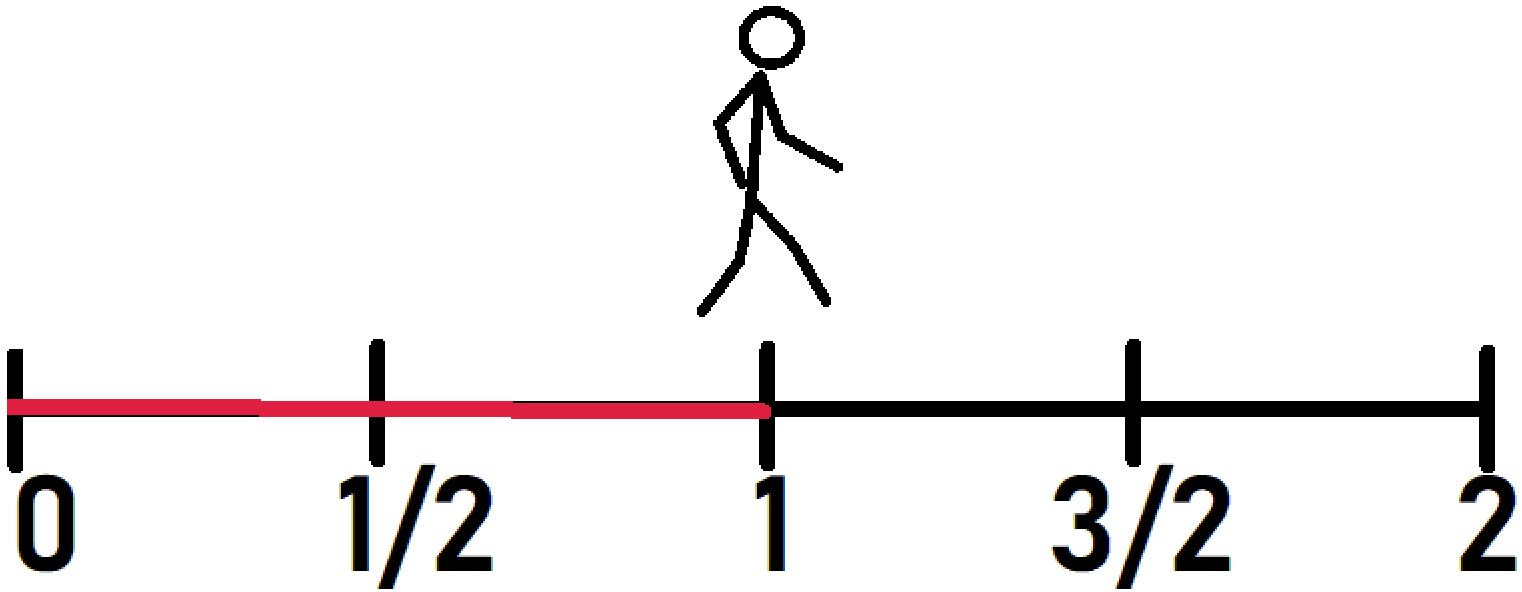

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The Geometric Series Formula

The Geometric Series Formula is one of the most important in mathematics. It is one of the few sums we can evaluate exactly.

Lemma: If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots + r^n = \frac{1 - r^{n+1}}{1-r}$.

Proof: Let $S_n = 1 + r + r^2 + r^3 + r^4 + \ldots + r^n$

Then $\quad r S_n = \quad r + r^2 + r^3 + r^4 + \ldots + r^n + r^{n+1}$

What should we do now?

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Then $\quad r\, S_n = \quad r + r^2 + r^3 + r^4 + \ldots + r^n + r^{n+1}$

Subtract: $S_n - r\, S_n = 1 - r^{n+1}$,

So $(1-r)\, S_n = 1 - r^{n+1}$, or S_n

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So $(1-r) S_n = 1 - r^{n+1}$, or $S_n = \frac{1 - r^{n+1}}{1-r}$.

If we let n go to infinity, we see r^{n+1} goes to 0, so we get the infinite sum is $\frac{1}{1-r}$.

Simpler Game: Hoops

Game of hoops: first basket wins, alternate shooting.

We will prove the Geometric Series Formula just by studying this basketball game!

Simpler Game: Hoops: Mathematical Formulation

Bird and **Magic** (I'm old!) alternate shooting; first basket wins.

- **Bird** always gets basket with probability p.
- **Magic** always gets basket with probability q.

Let x be the probability **Bird** wins – what is x?

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

Solving the Hoop Game

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Break into cases:

- **Bird** wins on 1^{st} shot: p.

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Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1 - p)(1 - q) \cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1-p)(1-q) \cdot p$.
- **Bird** wins on 3rd shot: $(1-p)(1-q) \cdot (1-p)(1-q) \cdot p$.

Solving the Hoop Game

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- **Bird** wins on 2nd shot: $(1-p)(1-q)\cdot p$.
- **Bird** wins on 3rd shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdot p$.
- **Bird** wins on nth shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

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Classic solution involves the geometric series.

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- **Bird** wins on nth shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Let $r=(1-p)(1-q)$. Then

$$\begin{aligned} x &= \text{Prob}(\textbf{Bird}\text{ wins}) \\ &= p + rp + r^2p + r^3p + \cdots \\ &= p\left(1 + r + r^2 + r^3 + \cdots\right), \end{aligned}$$

the geometric series.

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

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Have

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Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1-p)(1-q) * ???$$

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Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1 - p)(1 - q)x$$

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Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1-p)(1-q)x = p + rx.$$

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Thus

$$(1 - r)x = p \quad \text{or} \quad x = \frac{p}{1 - r}.$$

As $x = p(1 + r + r^2 + r^3 + \cdots)$, find

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1 - r}.$$

Advanced Geometric Series Comments

Always carefully look at what you did, and be explicit on what you proved.

The geometric series formula is:

If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots = \frac{1}{1-r}$.

We proved this when r = (1-p)(1-q), where p and q are the probabilities of making a basket for Bird and Magic. What are the ranges for p and q? We have what range of p and q?

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Lessons from Hoop Problem

- ◇ Power of Perspective: Memoryless process.
- ◇ Can circumvent algebra with deeper understanding! (Hard)
- ◇ Depth of a problem not always what expect.
- ◇ Importance of knowing more than the minimum: connections.
- ◇ Math is fun!

New Sum: The Harmonic Series

The Harmonic Series $\{H_n\}$ is defined as the sequence where

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Thus the first few terms are

- 1,
- 1 + 1/2 = 3/2 = 1.5,
- 1 + 1/2 + 1/3 = 11/6 or about 1.83,
- 1 + 1/2 + 1/3 + 1/4 = 25/12 or about 2.08
- $H_{100} = \frac{14\,466\,636\,279\,520\,351\,160\,221\,518\,043\,104\,131\,447\,711}{2\,788\,815\,009\,188\,499\,086\,581\,352\,357\,412\,492\,142\,272}$ or about 5.18
- H_{10000} is about 9.78
- $H_{1000000}$ is about 14.3927; the terms are growing but VERY slowly.....

The Harmonic Series Diverges!

The Harmonic Series $\{H_n\}$ is the sequence where $H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Let H be the limit as n goes to infinity of H_n, thus it is the sum of the reciprocals of integers. We claim H = ∞, *so the sum diverges*

Proof: Assume H is finite, let H_{even} be the sum of the reciprocals of even numbers, H_{odd} the sum of the odd terms.

$$H_{odd} = \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \quad H_{even} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$$

As 1/1 > 1/2, 1/3 > 1/4, what can you say about the size of H_{odd} versus the size of H_{even}?

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Proof: Assume H is finite, let H_{even} be the sum of the reciprocals of even numbers, H_{odd} the sum of the odd terms. As 1/1 > 1/2, 1/3 > 1/4, and so on we see the sum of the odd terms is larger than the sum of the evens.

Thus $H = H_{even} + H_{odd} > H_{even} + H_{even} = 2H_{even}$.

Note however that $H_{even} = 1/2 + 1/4 + 1/6 + 1/8 + \ldots = \frac{1}{2}(1 + 1/2 + 1/3 + 1/4 + \ldots) = \frac{1}{2}$ H.

Why is this true?

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The Harmonic Series $\{H_n\}$ is the sequence where $H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$.

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Proof: Assume H is finite, let H_{even} be the sum of the reciprocals of even numbers, H_{odd} the sum of the odd terms. As 1/1 > 1/2, 1/3 > 1/4, and so on we see the sum of the odd terms is larger than the sum of the evens.

Thus $H = H_{even} + H_{odd} > H_{even} + H_{even} = 2H_{even}$.

Note however that $H_{even} = 1/2 + 1/4 + 1/6 + 1/8 + \ldots = \frac{1}{2}(1 + 1/2 + 1/3 + 1/4 + \ldots) = \frac{1}{2} H$.

So $H > 2 H_{even} = 2 * \frac{1}{2} H = H$; why is this a contradiction?

The Harmonic Series Diverges!

The Harmonic Series $\{H_n\}$ is the sequence where $H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$.

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Note however that $H_{even} = 1/2 + 1/4 + 1/6 + 1/8 + \ldots = \frac{1}{2}(1 + 1/2 + 1/3 + 1/4 + \ldots) = \frac{1}{2}$ H.

So $H > 2\ H_{even} = 2 * \frac{1}{2}\ H = H$; but H cannot be larger than H, contradiction, thus our assumption that H converges is false!

The Harmonic Series Diverges!

The Harmonic Series $\{H_n\}$ is the sequence where $H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$.

The divergence of this sum is so important we give another proof.

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \cdots$$

If we group terms together, we can get infinitely many sums that are more than 1/2, so it diverges.

What should we group with 1/3 to get terms that sum to more than 1/2?

The Harmonic Series Diverges!

The Harmonic Series $\{H_n\}$ is the sequence where $H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$.

The divergence of this sum is so important we give another proof.

$$\frac{1}{1} + \frac{1}{2} + \boxed{\frac{1}{3} + \frac{1}{4}} + \boxed{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \boxed{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}} + \cdots$$

If we group terms together, we can get infinitely many sums that are more than 1/2, so it diverges.

Note 1/3 and 1/4 are each at least 1/4, so their sum is at least 2 * 1/2 = 1/2.

Note 1/5, …, 1/8 are each at least 1/8, so their sum is at least 4 * 1/8 = 1/2.

Note 1/9, …, 1/16 are each at least 1/16, so their sum is at least 8 * 1/16 = 1/2.

Part III: From the Geometric Series Formula to Primes

JOURNAL OF NUMBER THEORY

Teachers As Scholars
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Application of the Geometric Series Formula: Infinitude of Primes!

One of the most important applications of the Geometric Series Formula is in Number Theory.

It is used in creating / understanding the Riemann Zeta Function, which gives us tremendous information about primes.

Remember primes are numbers with exactly two factors, 1 and themselves: 2, 3, 5, 7, 11, 13, 17, 19, 23, …. If you are divisible by two or more primes you are called composite, while 1 is called a unit. We will see it is convenient NOT to have 1 be a prime.

Euclid and the Infinitude of Primes

There are many proofs that there are infinitely many primes. This one goes back over 2000 years to Euclid….

Assume there are only finitely many primes, say $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, …, p_n.

Consider the new number $x = p_1 * p_2 * p_3 * \ldots * p_n + 1$.

Can this be divisible by p_1?

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Assume there are only finitely many primes, say $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, …, p_n.

Consider the new number $x = p_1 * p_2 * p_3 * \ldots * p_n + 1$.

Can this be divisible by p_1? No, the remainder is 1.

Can this be divisible by p_2?

Euclid and the Infinitude of Primes

There are many proofs that there are infinitely many primes. This one goes back over 2000 years to Euclid….

Assume there are only finitely many primes, say $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, …, p_n.

Consider the new number $x = p_1 * p_2 * p_3 * \ldots * p_n + 1$.
Can this be divisible by p_1? No, the remainder is 1.
Can this be divisible by p_2? No, the remainder is 1.

Continuing we see it cannot be divisible by ANY prime in our list. As we assumed our list was complete, we have found a new prime (either this number is prime, or it is divisible by a prime not on our list).

Euclid and the Infinitude of Primes

Consider the numbers generated by Euclid’s method; it’s fun to try this process.

- We start with 2, then look at 2+1 and get 3 as the next number.
- Then $2 * 3 + 1 = 7$ for our next prime.
- Then $2 * 3 * 7 + 1 = 43$ which is also prime.

Do we always get a prime when we apply this? Do we get all the primes?

Euclid and the Infinitude of Primes

Consider the numbers generated by Euclid's method; It's fun to try this process.

- We start with 2, then look at 2+1 and get 3 as the next number.
- Then 2 * 3 + 1 = 7 for our next prime.
- Then 2 * 3 * 7 + 1 = 43 which is also prime.

Do we always get a prime when we apply this? Do we get all the primes?

We do not always get a prime – look at the next term!

- 2 * 3 * 7 * 43 + 1 = 1807 = 13 * 139.

The other questions are open….. We don't have to go far to find open questions about primes (others include are there infinitely many pairs of primes differing by 2, and can every even number at least 4 be written as the sum of two primes).

https://en.wikipedia.org/wiki/Euclid–Mullin_sequence

Euclid–Mullin sequence

From Wikipedia, the free encyclopedia

The **Euclid–Mullin sequence** is an infinite sequence of distinct prime numbers, in which each element is the least prime factor of one plus the product of all earlier elements. They are named after the ancient Greek mathematician Euclid, because their definition relies on an idea in Euclid's proof that there are infinitely many primes, and after Albert A. Mullin, who asked about the sequence in 1963.[1]

The first 51 elements of the sequence are

> 2, 3, 7, 43, 13, 53, 5, 6221671, 38709183810571, 139, 2801, 11, 17, 5471, 52662739, 23003, 30693651606209, 37, 1741, 1313797957, 887, 71, 7127, 109, 23, 97, 159227, 643679794963466223081509857, 103, 1079990819, 9539, 3143065813, 29, 3847, 89, 19, 577, 223, 139703, 457, 9649, 61, 4357, 87991098722552272708281251793312351581099392851768893748012603709343, 107, 127, 3313, 227432689108589532754984915075774848386671439568260420754414940780761245893, 59, 31, 211... (sequence A000945 in the OEIS)

These are the only known elements as of September 2012. Finding the next one requires finding the least prime factor of a 335-digit number (which is known to be composite).

The Riemann Zeta Function ζ(s)

https://en.wikipedia.org/wiki/Greek_alphabet

Greek alphabet

From Wikipedia, the free encyclopedia

The **Greek alphabet** has been used to write the Greek language since the late ninth or early eighth century BC.[3][4] It is derived from the earlier Phoenician alphabet,[5] and was the first alphabetic script in history to have distinct letters for vowels as well as consonants. In Archaic and early Classical times, the Greek alphabet existed in many different local variants, but, by the end of the fourth century BC, the Euclidean alphabet, with twenty-four letters, ordered from alpha to omega, had become standard and it is this version that is still used to write Greek today. These twenty-four letters (each in uppercase and lowercase forms) are: Α α, Β β, Γ γ, Δ δ, Ε ε, Ζ ζ, Η η, Θ θ, Ι ι, Κ κ, Λ λ, Μ μ, Ν ν, Ξ ξ, Ο ο, Π π, Ρ ρ, Σ σ/ς, Τ τ, Υ υ, Φ φ, Χ χ, Ψ ψ, and Ω ω.

ζ(s)

There are many different ways of writing a Greek letter zeta; here is how Powerpoint displays it.

The Riemann Zeta Function ζ(s)

We define this function as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \ldots$$

and for us we will take s > 1 which ensures the infinite sum converges (for those knowing more, s can be any complex number with real part at least 1).

Looking at this function, it is NOT clear why it is worth studying….

Integers and Primes

Most of us are familiar with the positive integers: 1, 2, 3, 4, 5, ….

What is the next integer after 2020?

Integers and Primes

Most of us are familiar with the positive integers: 1, 2, 3, 4, 5, ….

What is the next integer after 2020? 2021

What is the next integer after 2021?

Integers and Primes

Most of us are familiar with the positive integers: 1, 2, 3, 4, 5, ….

What is the next integer after 2020? 2021

What is the next integer after 2021? 2022

What is the next integer after 2022?

Integers and Primes

Most of us are familiar with the positive integers: 1, 2, 3, 4, 5, ….

What is the next integer after 2020? 2021

What is the next integer after 2021? 2022

What is the next integer after 2022? 2023

As you have hopefully noticed, there is not much mystery in the spacings between integers!

Integers and Primes

What about the primes: 2, 3, 5, 7, ….

What is the next prime after 2020?

Integers and Primes

What about the primes: 2, 3, 5, 7, ...

What is the next prime after 2020? 2027

What is the next prime after 2027?

Integers and Primes

What about the primes: 2, 3, 5, 7, ….

What is the next prime after 2020? 2027

What is the next prime after 2027? 2029

What is the next prime after 2029?

Integers and Primes

What about the primes: 2, 3, 5, 7, ….

What is the next prime after 2020? 2027

What is the next prime after 2027? 2029

What is the next prime after 2029? 2039

As you have hopefully noticed, it is a lot harder to find the next prime than to find the next integer!

The Riemann Zeta Function ζ(s) and Primes

We defined the Riemann Zeta Function (for s > 1) by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \ldots$$

and now we note a remarkable property; we also have

$$\zeta(s) = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1} = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$$

Two questions: (1) Why is this true, and (2) Why do we care?

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

Why do we care?

The integers are completely understood. We even have a great formula for the n^{th} integer!

The Riemann zeta function connects the integers and the primes.

Perhaps we can pass from knowledge about the integers to knowledge about the primes….

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

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If we take s=1 the sum becomes the Harmonic Series, which we showed diverges!

If there were only finitely many primes the product would ???.

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

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If we take s=1 the sum becomes the Harmonic Series, which we showed diverges!

If there were only finitely many primes the product would converge!

Thus there are infinitely many primes! (Advanced: can prove more, can prove the sum of the reciprocals of the primes diverges.)

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

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The following is beyond the scope of this talk, but if we take s=2 then the sum is $\pi^2 / 6$, which is an irrational number (this means we cannot write it as a ratio of two integers).

If there were only finitely many primes then the product would be a finite product of rational numbers, and hence rational! For example, if only 2 and 3 are prime:

$$\prod_{p\ prime} \left(1 - \frac{1}{p^2}\right)^{-1} = \left(1 - \frac{1}{2^2}\right)^{-1} \left(1 - \frac{1}{3^2}\right)^{-1} = \left(\frac{3}{4}\right)^{-1} \left(\frac{8}{9}\right)^{-1} = \frac{4}{3}\frac{9}{8} = \frac{3}{2}$$

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

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Thus there are infinitely many primes!

The Riemann Zeta Function ζ(s) and Primes

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We thus see the importance of the formula above, which connects sums over integers with products over primes.

It allows us to pass from knowledge of integers to knowledge of primes.

We now prove it, or at least sketch the proof.

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \cdots.$

We need the Fundamental Theorem of Arithmetic: Every positive integer can be written uniquely as a product of prime powers, where we write the primes in increasing order, and we let the empty product be 1.

Thus 12 =

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

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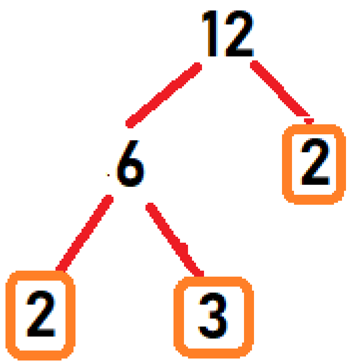
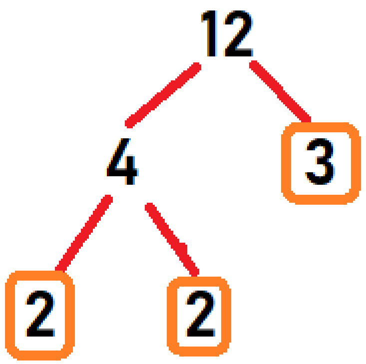
We need the Fundamental Theorem of Arithmetic: Every positive integer can be written uniquely as a product of prime powers, where we write the primes in increasing order, and we let the empty product be 1.

Thus $12 = 2^2 * 3$





and 90 =

The Riemann Zeta Function ζ(s) and Primes

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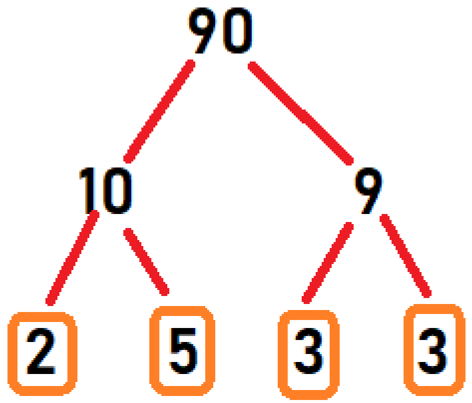
or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ldots .$

We need the Fundamental Theorem of Arithmetic: Every positive integer can be written uniquely as a product of prime powers, where we write the primes in increasing order, and we let the empty product be 1.



Thus $12 = 2^2 * 3$ and $90 = 2 * 3^2 * 5$,

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

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We need the Fundamental Theorem of Arithmetic: Every positive integer can be written uniquely as a product of prime powers, where we write the primes in increasing order, and we let the empty product be 1.

Thus $12 = 2^2 * 3$ and $90 = 2 * 3^2 * 5$, and there are no other ways to write these numbers. If 1 were prime, we would lose uniqueness: $2^2 * 3 = 1^{2020} * 2^2 * 3$.

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

We will not give a fully rigorous argument.

What we do is consider a finite product, the product over the first P primes, and show that as P gets larger and larger we get more and more of the terms in the sum (once and only once), including all the terms up to P, and thus in the limit as we take all the primes we get the sum.

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

We use the Geometric Series Formula to expand each factor.

$\left(1 - \frac{1}{p^s}\right)^{-1} = \frac{1}{\left(1 - \frac{1}{p^s}\right)}$ and this is a Geometric Series with $r = 1/p^s$.

Since $1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}$, we have $\left(1 - \frac{1}{p^s}\right)^{-1} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots$

The Riemann Zeta Function ζ(s) and Primes

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We use the Geometric Series Formula to expand each factor. If p = 2:

$\left(1 - \frac{1}{2^s}\right)^{-1} = \frac{1}{\left(1 - \frac{1}{2^s}\right)}$ and this is a Geometric Series with r = $1/p^s$.

Since $1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}$, we have $\left(1 - \frac{1}{2^s}\right)^{-1} = 1 + \frac{1}{2^s} + \frac{1}{(2^2)^s} + \frac{1}{(2^3)^s} + \ldots$

But $(2^2)^s$ = ???^{s}, $(2^3)^s$ = ??? ^{s}, so….

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

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Since $1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}$, we have $\left(1 - \frac{1}{2^s}\right)^{-1} = 1 + \frac{1}{2^s} + \frac{1}{(2^2)^s} + \frac{1}{(2^3)^s} + \ldots$

But $(2^2)^s = 4^s$, $(2^3)^s = 8^s$, so....

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

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$\left(1 - \frac{1}{2^s}\right)^{-1} = \frac{1}{\left(1 - \frac{1}{2^s}\right)}$ and this is a Geometric Series with r = $1/p^s$.

Since $1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}$, we have $\left(1 - \frac{1}{2^s}\right)^{-1} = 1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \ldots$

since $(2^2)^s = 4^s$, $(2^3)^s = 8^s$, ….

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

Let's look at multiplying the factors

$$\left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} = \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \ldots\right) * \left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{27^s} + \ldots\right)$$

When we multiply out we get

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{1}{12^s} + \frac{1}{16^s} + \frac{1}{18^s} + \frac{1}{24^s} + \frac{1}{27^s} + \frac{1}{32^s} + \frac{1}{36^s} + \cdots$$

We get exactly the numbers that have only 2 and 3 as prime factors....

Conclusion

Primes are the building blocks of numbers.

There are many questions about them, and most are beyond our ability to answer!

Algebra I is useful – helps to be able to expand products, to deal with exponents, ….

We need tools to approach them.

We saw how to use our knowledge of sums like the Geometric Series and the Harmonic Series to learn about the primes; what allows us to do this is the Riemann Zeta Function, which translates information about the integers (which are well understood) to information about the primes.