Tests of the L-Functions Ratios Conjecture

Steven J Miller
(joint work with Duc Khiem Huynh,
David Montague and Ralph Morrison)
Williams College

Steven.J.Miller@williams.edu
http://www.williams.edu/go/math/sjmiller/

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History

- Farmer (1993): Considered

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\,dt,$$

conjectured (for appropriate values)

$$T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}.$$

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- Conrey-Farmer-Zirnbauer (2007): conjecture formulas for averages of products of L-functions over families:

$$R_{\mathcal{F}} = \sum_{f\in\mathcal{F}} \omega_f \frac{L\left(\frac{1}{2}+\alpha, f\right)}{L\left(\frac{1}{2}+\gamma, f\right)}.$$

Uses of the Ratios Conjecture

- Applications:
  - ◇ n-level correlations and densities;
  - ◇ mollifiers;
  - ◇ moments;
  - ◇ vanishing at the central point.

- Advantages:
  - ◇ RMT models often add arithmetic ad hoc;
  - ◇ Predicts lower order terms to square-root level;
  - ◇ Fast computations.

Inputs for 1-level density

- Approximate Functional Equation:

$$L(s, f) = \sum_{m \leq x} \frac{a_m}{m^s} + \epsilon \mathbb{X}_L(s) \sum_{n \leq y} \frac{a_n}{n^{1-s}} + \text{Error};$$

◇ ϵ sign of the functional equation,
◇ $\mathbb{X}_L(s)$ ratio of Γ-factors from functional equation.

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- Explicit Formula: g Schwartz test function,

$$\sum_{f\in\mathcal{F}} \omega_f \sum_{\gamma} g\left(\gamma \frac{\log N_f}{2\pi}\right) = \frac{1}{2\pi i} \int_{(c)-(1-c)} R'_{\mathcal{F}}(\cdots) g(\cdots)$$

  ◇ $R'_{\mathcal{F}}(r) = \frac{\partial}{\partial\alpha} R_{\mathcal{F}}(\alpha,\gamma)\Big|_{\alpha=\gamma=r}$.

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where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.

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- Differentiate with respect to the parameters.

Sympletic Results

Symplectic Families

- Fundamental discriminants: d square-free and 1 modulo 4, or $d/4$ square-free and 2 or 3 modulo 4.
- Associated character χ_d:
  ◊ $\chi_d(-1) = 1$ say d even;
  ◊ $\chi_d(-1) = -1$ say d odd.
  ◊ even (resp., odd) if $d > 0$ (resp., $d < 0$).

Will study following families:

◊ even fundamental discriminants at most X;
◊ $\{8d : 0 < d \leq X, \ d$ an odd, positive square-free fundamental discriminant$\}$.

Prediction from Ratios Conjecture

$$\begin{aligned}\frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_d} g\left(\gamma_d\frac{\log X}{2\pi}\right) &= \frac{1}{X^*\log X}\int_{-\infty}^{\infty} g(\tau)\sum_{d\le X}\left[\log\frac{d}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\pm\frac{i\pi\tau}{\log X}\right)\right]d\tau \\ &+ \frac{2}{X^*\log X}\sum_{d\le X}\int_{-\infty}^{\infty} g(\tau)\left[\frac{\zeta'}{\zeta}\left(1+\frac{4\pi i\tau}{\log X}\right)+A_D'\left(\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)\right. \\ &\left. - e^{-2\pi i\tau\log(d/\pi)/\log X}\frac{\Gamma\left(\frac{1}{4}-\frac{\pi i\tau}{\log X}\right)}{\Gamma\left(\frac{1}{4}+\frac{\pi i\tau}{\log X}\right)}\zeta\left(1-\frac{4\pi i\tau}{\log X}\right)A_D\left(-\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)\right]d\tau + O(X^{-\frac{1}{2}+\epsilon}),\end{aligned}$$

with

$$A_D(-r,r) = \prod_p\left(1-\frac{1}{(p+1)p^{1-2r}}-\frac{1}{p+1}\right)\cdot\left(1-\frac{1}{p}\right)^{-1}$$

$$A_D'(r;r) = \sum_p\frac{\log p}{(p+1)(p^{1+2r}-1)}.$$

Proof: Contour shifts, $A_D(-r;r)=\zeta(2)/\zeta(2-2r)$.

Prediction from Ratios Conjecture (cont)

Main term is

$$\frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_d} g\left(\gamma_d\frac{\log X}{2\pi}\right) = \int_{-\infty}^{\infty} g(x)\left(1-\frac{\sin(2\pi x)}{2\pi x}\right)dx + O\left(\frac{1}{\log X}\right),$$

which is the 1-level density for the scaling limit of $\mathrm{USp}(2N)$. If $\mathrm{supp}(\widehat{g})\subset(-1,1)$, then the integral of $g(x)$ against $-\sin(2\pi x)/2\pi x$ is $-g(0)/2$.

Prediction from Ratios Conjecture (cont)

Assuming RH for $\zeta(s)$, for $\mathrm{supp}(\widehat{g}) \subset (-\sigma,\sigma) \subset (-1,1)$:

$$\frac{-2}{X^* \log X} \sum_{d \le X} \int_{-\infty}^{\infty} g(\tau)\, e^{-2\pi i \tau \frac{\log(d/\pi)}{\log X}} \frac{\Gamma\left(\frac14 - \frac{\pi i \tau}{\log X}\right)}{\Gamma\left(\frac14 + \frac{\pi i \tau}{\log X}\right)} \zeta\left(1 - \frac{4\pi i \tau}{\log X}\right) A_D\left(-\frac{2\pi i \tau}{\log X}; \frac{2\pi i \tau}{\log X}\right) d\tau$$

$$= -\frac{g(0)}{2} + O(X^{-\frac34(1-\sigma)+\epsilon});$$

error term absorbed into $O(X^{-1/2+\epsilon})$ if $\sigma < 1/3$.

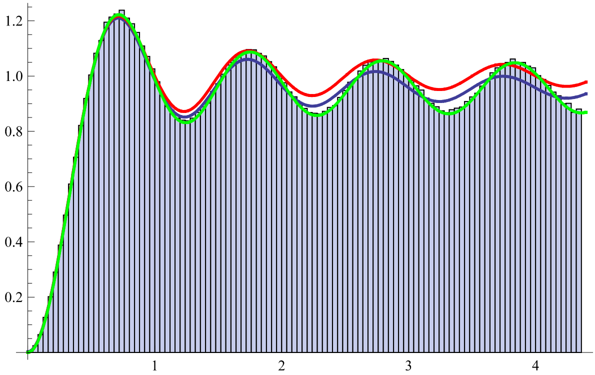
Main Results

Theorem (M– '07)

Let $\text{supp}(\widehat{g}) \subset (-\sigma, \sigma)$, *assume RH for* $\zeta(s)$. 1-*Level Density agrees with prediction from Ratios Conjecture*

- *up to* $O(X^{-(1-\sigma)/2+\epsilon})$ *for the family of quadratic Dirichlet characters with even fundamental discriminants at most* X*;*
- *up to* $O(X^{-1/2} + X^{-(1-\frac{3}{2}\sigma)+\epsilon} + X^{-\frac{3}{4}(1-\sigma)+\epsilon})$ *for our sub-family. If* $\sigma < 1/3$ *then agrees up to* $O(X^{-1/2+\epsilon})$.

Numerics (J. Stopple): 1,003,083 negative fundamental discriminants $-d \in [10^{12}, 10^{12} + 3.3 \cdot 10^6]$



Histogram of normalized zeros ($\gamma \le 1$, about 4 million).
$\diamond$ Red: main term. $\diamond$ Blue: includes $O(1/\log X)$ terms.
$\diamond$ Green: all lower order terms.

Sketch of
Symplectic Proofs

Ratios Calculation

Hardest piece to analyze is

$$R(g;X) = -\frac{2}{X^* \log X} \sum_{d \le X} \int_{-\infty}^{\infty} g(\tau) e^{-2\pi i \tau \frac{\log(d/\pi)}{\log X}} \frac{\Gamma\left(\frac{1}{4} - \frac{\pi i \tau}{\log X}\right)}{\Gamma\left(\frac{1}{4} + \frac{\pi i \tau}{\log X}\right)} \cdot \zeta\left(1 - \frac{4\pi i \tau}{\log X}\right) A_D\left(-\frac{2\pi i \tau}{\log X}; \frac{2\pi i \tau}{\log X}\right) d\tau,$$

$$A_D(-r,r) = \prod_p \left(1 - \frac{1}{(p+1)p^{1-2r}} - \frac{1}{p+1}\right) \cdot \left(1 - \frac{1}{p}\right)^{-1}.$$

Proof: shift contours, keep track of poles of ratios of Γ and zeta functions, $A_D(-r;r) = \zeta(2)/\zeta(2-2r)$.

Ratios Calculation: Weaker result for $\text{supp}(\widehat{g}) \subset (-1, 1)$.

- d-sum is $X^* e^{-2\pi i \left(1 - \frac{\log \pi}{\log X}\right)\tau} \left(1 - \frac{2\pi i \tau}{\log X}\right)^{-1} + O(X^{1/2})$;

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- decay of g restricts τ-sum to $|\tau| \le \log X$, Taylor expand everything but g: small error term and

$$\int_{|\tau|\le \log X} g(\tau) \sum_{n=-1}^{N} \frac{a_n}{\log^n X} (2\pi i \tau)^n e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau} d\tau$$

$$= \sum_{n=-1}^{N} \frac{a_n}{\log^n X} \int_{|\tau|\le \log X} (2\pi i \tau)^n g(\tau) e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau} d\tau;$$

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- from decay of g can extend the τ-integral to $\mathbb{R}$ (essential that N is fixed and finite!), for $n \ge 0$ get the Fourier transform of $g^{(n)}$ (the n^{th} derivative of g) at $1-\frac{\pi}{\log X}$, vanishes if $\operatorname{supp}(\widehat{g}) \subset (-1,1)$.

Number Theory Sums

$$S_{\text{even}} = -\frac{2}{X^*}\sum_{d\le X}\sum_{\ell=1}^{\infty}\sum_{p}\frac{\chi_d(p)^2\log p}{p^\ell\log X}\,\widehat{g}\left(2\frac{\log p^\ell}{\log X}\right)$$

$$S_{\text{odd}} = -\frac{2}{X^*}\sum_{d\le X}\sum_{\ell=0}^{\infty}\sum_{p}\frac{\chi_d(p)\log p}{p^{(2\ell+1)/2}\log X}\,\widehat{g}\left(\frac{\log p^{2\ell+1}}{\log X}\right).$$

Number Theory Sums

Lemma

Let $\mathrm{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$. *Then*

$$\begin{aligned} S_{\mathrm{even}} &= -\frac{g(0)}{2} + \frac{2}{\log X} \int_{-\infty}^{\infty} g(\tau) \frac{\zeta'}{\zeta} \left(1 + \frac{4\pi i \tau}{\log X}\right) d\tau \\ &\quad + \frac{2}{\log X} \int_{-\infty}^{\infty} g(\tau) A_D' \left(\frac{2\pi i \tau}{\log X}; \frac{2\pi i \tau}{\log X}\right) + O(X^{-\frac{1}{2}+\epsilon}) \\ S_{\mathrm{odd}} &= O(X^{-\frac{1-\sigma}{2}} \log^6 X). \end{aligned}$$

If instead we consider the family of characters χ_{8d} *for odd, positive square-free* $d \in (0, X)$ *(*d *a fundamental discriminant), then*

$$S_{\mathrm{odd}} = O(X^{-1/2+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon}).$$

Analysis of S_{even}

$\chi_d(p)^2 = 1$ except when $p|d$. Replace $\chi_d(p)^2$ with 1, and subtract off the contribution from when $p|d$:

$$\begin{aligned} S_{\text{even}} &= -2\sum_{\ell=1}^{\infty}\sum_{p}\frac{\log p}{p^{\ell}\log X}\,\widehat{g}\left(2\frac{\log p^{\ell}}{\log X}\right) \\ &\quad + \frac{2}{X^*}\sum_{d\le X}\sum_{\ell=1}^{\infty}\sum_{p|d}\frac{\log p}{p^{\ell}\log X}\,\widehat{g}\left(2\frac{\log p^{\ell}}{\log X}\right) \\ &= S_{\text{even};1} + S_{\text{even};2}. \end{aligned}$$

Lemma (Perron's Formula)

$$S_{\text{even};1} = -\frac{g(0)}{2} + \frac{2}{\log X}\int_{-\infty}^{\infty} g(\tau)\frac{\zeta'}{\zeta}\left(1+\frac{4\pi i\tau}{\log X}\right)d\tau.$$

Analysis of S_{even}: $S_{\text{even};2}$

This piece gives us $\int g(\tau) A_D'(-\cdots, \cdots)$.

- Main ideas:
  - ⋄ Restrict to $p \le X^{1/2}$.
  - ⋄ For $p < X^{1/2}$: $\sum_{d \le X, p|d} 1 = \frac{X^*}{p+1} + O(X^{1/2})$.
  - ⋄ Use Fourier Transform to expand $\widehat{g}$.

Analysis of S_{odd}

$$S_{\text{odd}} = -\frac{2}{X^*}\sum_{\ell=0}^{\infty}\sum_{p}\frac{\log p}{p^{(2\ell+1)/2}\log X}\,\widehat{g}\left(\frac{\log p^{2\ell+1}}{\log X}\right)\sum_{d\le X}\chi_d(p).$$

Analysis of $S_{\rm odd}$

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Jutila's bound

$$\sum_{\substack{1<n\le N\\ n \text{ non-square}}}\left|\sum_{\substack{0<d\le X\\ d \text{ fund. disc.}}}\chi_d(n)\right|^2 \ll NX\log^{10}N.$$

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Proof: Cauchy-Schwarz and Jutila: $p^{2\ell+1}$ non-square:

$$\left(\sum_{\ell=0}^{\infty}\sum_{p^{(2\ell+1)/2}\le X^\sigma}\left|\sum_{d\le X}\chi_d(p)\right|^2\right)^{1/2} \ll X^{\frac{1+\sigma}{2}}\log^5 X.$$

Analysis of $S_{\rm odd}$: Extending Support

More technical, replace Jutila's bound by applying Poisson Summation to character sums (Gao's thesis, Michigan 2005).

Lemma

Let $\mathrm{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$*. For family* $\{8d\ :\ 0 < d \le X,\ d$ *an odd, positive square-free fundamental discriminant*$\}$*,* $S_{\rm odd} = O(X^{-\frac{1}{2}+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon})$*. In particular, if* $\sigma < 1/3$ *then* $S_{\rm odd} = O(X^{-1/2+\epsilon})$*.*

Orthogonal Results
(joint with David Montague)

Background

Study $L(s,f) = \sum \lambda_f(n) n^{-s}$ with f ranging over cuspidal newforms of weight k and prime level $N \to \infty$.

Iwaniec-Luo-Sarnak calculated 1-level density if $\mathrm{supp}(\widehat{\phi}) \subset (-2, 2)$.

Key ingredient: averaging $\lambda_f(n)$'s over family by the Petersson formula.

Note: Use harmonic weights and assume level N prime to facilitate using Petersson formula.

Petersson Formula

Let

$$\Delta_{k,N}(m,n) \ = \ \sum_{f\in\mathcal{B}_k(N)} \omega_f(N)\lambda_f(m)\lambda_f(n).$$

We have

$$\Delta_{k,N}(m,n) \ = \ \delta(m,n) + 2\pi i^k \sum_{c\equiv 0 \bmod N} \frac{S(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

where $\delta(m,n)$ is the Kronecker symbol

$$S(m,n;c) \ = \ \sum_{d \bmod c}^{*} \exp\left(2\pi i \frac{md+n\overline{d}}{c}\right)$$

is the classical Kloosterman sum ($d\overline{d} \equiv 1 \bmod c$), and $J_{k-1}(x)$ is a Bessel function.

Consequences of the Petersson Formula

$$\Delta_{k,N}(m,n) \;=\; \delta(m,n) + 2\pi i^k \sum_{c \equiv 0 \bmod N} \frac{S(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

The Bessel-Kloosterman piece contributes an error term if $\sigma < 1$ and a main term otherwise.

The ‘diagonal’ piece does not include the Bessel-Kloosterman term, which we know contributes!

Possible danger: Ratios Conjecture says only to keep diagonal or main terms, and dropping a smaller contribution which becomes quite large!

Main Results: Test for family $\mathcal{F} = H_k^{\pm}(N)$

This family is an important test: the non-diagonal terms that are dropped contribute to the main term!

Theorem: Ratios Conjecture Prediction (M-Montague)

With $\chi(s) = \prod_p \left(1 + \frac{1}{(p-1)p^s}\right)$, the 1-level density is

$$\sum_p \frac{2\log p}{p\log R}\widehat{\phi}\left(\frac{2\log p}{\log R}\right)$$

$$\mp 2\lim_{\epsilon\downarrow 0}\int_{-\infty}^{\infty}\mathbb{X}_L\left(\frac{1}{2}+2\pi ix\right)\chi(\epsilon+4\pi ix)\phi(t\log R)dt$$

$$-\int_{-\infty}^{\infty}\frac{\mathbb{X}_L'}{\mathbb{X}_L}\left(\frac{1}{2}+2\pi it\right)\phi(t\log R)dt + O(N^{-1/2+\epsilon}).$$

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Theorem: Agreement with Number Theory (M-Montague)

Assume GRH for $\zeta(s)$, Dirichlet L-functions, and $L(s, f)$. For ϕ such that $\text{supp}(\widehat{\phi}) \subset (-1, 1)$, the 1-level density agrees with the ratios conjecture prediction up to $O(N^{-1/2+\epsilon})$, and get agreement up to a power savings in N if $\text{supp}(\widehat{\phi}) \subset (-2, 2)$.

Key fact

Theorem:

For $\mathfrak{Re}(\alpha), \mathfrak{Re}(\gamma) > 0$, the Ratios Conjecture predicts that

$$\begin{aligned}
\mathcal{R}_{\pm}(N) \quad &:= \quad \sum_{f \in H_k^{\pm}(N)} \omega_f^{\pm}(N) \frac{L(\frac{1}{2}+\alpha, f)}{L(\frac{1}{2}+\gamma, f)} \\
&= \quad \prod_p \left(1 - \frac{1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}}\right) \pm \mathbb{X}_L\left(\frac{1}{2}+\alpha\right) \\
&\quad \cdot \frac{1}{\zeta(1-\alpha+\gamma)} \prod_p \left(1 + \frac{p^{1-\alpha+\gamma}}{p^{1+2\gamma}(p^{1-\alpha+\gamma}-1)}\right) \\
&\quad + O(N^{-1/2+\epsilon}).
\end{aligned}$$

Proof (cont)

Proof: $\mathcal{R}_{\pm}(N)$ equals

$$\sum_{f\in H_k^*(N)} (1\pm\epsilon_f)\omega_f^*(N)\left(\sum_{h=1}^{\infty}\frac{\mu_f(h)}{h^{\frac{1}{2}+\gamma}}\right)\left[\sum_{m\leq x}\frac{\lambda_f(m)}{m^{\frac{1}{2}+\alpha}}+\epsilon_f\mathbb{X}_L\left(\frac{1}{2}+\alpha\right)\sum_{n\leq y}\frac{\lambda_f(n)}{n^{\frac{1}{2}-\alpha}}\right].$$

Expanding, find $\mathcal{R}_{\pm}(N)$ is

$$\sum_{f\in H_k^*(N)} \omega_f^*(N)\sum_{h=1}^{\infty}\frac{\mu_f(h)}{h^{\frac{1}{2}+\gamma}}\left[\sum_{m\leq x}\frac{\lambda_f(m)}{m^{\frac{1}{2}+\alpha}}+\epsilon_f\mathbb{X}_L\left(\frac{1}{2}+\alpha\right)\sum_{n\leq y}\frac{\lambda_f(n)}{n^{\frac{1}{2}-\alpha}}\right]$$
$$\pm\sum_{f\in H_k^*(N)} \omega_f^*(N)\sum_{h=1}^{\infty}\frac{\mu_f(h)}{h^{\frac{1}{2}+\gamma}}\left[\epsilon_f\sum_{m\leq x}\frac{\lambda_f(m)}{m^{\frac{1}{2}+\alpha}}+\mathbb{X}_L\left(\frac{1}{2}+\alpha\right)\sum_{n\leq y}\frac{\lambda_f(n)}{n^{\frac{1}{2}-\alpha}}\right].$$

Terms involving ϵ_f are negligible and may be dropped (part of Ratios Conjecture, but can prove small).

Proof (cont)

Left with

$$S_1 := \sum_{f\in H_k^*(N)} \omega_f^*(N) \sum_{h=1}^{\infty} \frac{\mu_f(h)}{h^{\frac12+\gamma}} \sum_{m\le x} \frac{\lambda_f(m)}{m^{\frac12+\alpha}}$$

$$S_2 := \pm \sum_{f\in H_k^*(N)} \omega_f^*(N) \sum_{h=1}^{\infty} \frac{\mu_f(h)}{h^{\frac12+\gamma}} \mathbb{X}_L\left(\frac12+\alpha\right) \sum_{n\le y} \frac{\lambda_f(n)}{n^{\frac12-\alpha}}.$$

Analysis yields

$$S_1 = \prod_p \left(1 - \frac{1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}}\right) + O(N^{-1/2+\epsilon})$$

$$S_2 = \pm \mathbb{X}_L\left(\frac12+\alpha\right) \frac{1}{\zeta(1-\alpha+\gamma)} \prod_p \left(1 + \frac{p^{1-\alpha+\gamma}}{p^{1+2\gamma}(p^{1-\alpha+\gamma}-1)}\right) + O(N^{-1/2+\epsilon}).$$

Proof (cont)

Differentiating yields

$$\sum_{f\in H_k^{\pm}(N)} \omega_f^{\pm}(N)\frac{L'(\frac{1}{2}+r,f)}{L(\frac{1}{2}+r,f)} = \sum_p \left(\frac{\log p}{p^{1+2r}}\right)\mp\mathbb{X}_L\left(\frac{1}{2}+r\right)\chi(2r)+O(N^{-1/2+\epsilon}),$$

where $\chi(s)$ is defined as

$$\chi(s) := \prod_p \left(1+\frac{1}{(p-1)p^s}\right).$$

Quadratic Twist Families (work in progress)
(joint with Duc Khiem Huynh, Ralph Morrison)

Families

1. Studying quadratic twists of a fixed elliptic curve (with Duc Khiem Huynh);
2. Studying quadratic twists of the τ function (with Ralph Morrison).

Second family easier (all primes are good).

Difficulty: analyzing product over prime piece.

Predictions

Number Theory Predictions:

$$\sum_{d\in\mathcal{F}(X)}\sum_{\gamma_d} g\left(\gamma_d\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\nu)\Bigg(\sum_{d\in\mathcal{F}(X)}\left[2\log\left(\frac{d}{2\pi}\right)+\frac{\Gamma'}{\Gamma}(6+i\nu)+\frac{\Gamma'}{\Gamma}(6-i\nu)\right]$$

$$+2\left(-\sum_p\sum_{k=1}^{\infty}\frac{(\alpha_p^{2k}+\overline{\alpha}_p^{2k})\log p}{p^{k(1+2i\nu)}}+\sum_p\frac{\log p}{(p+1)}\sum_{k=1}^{\infty}\frac{(\alpha_p^{2k}+\overline{\alpha}_p^{2k})}{p^{k(1+2i\nu)}}\right)\Bigg)d\nu$$

$$+O(X^{1/2}\log\log X).$$

Predictions

Ratios' Prediction:

$$\sum_{d\in\mathcal{F}(X)}\sum_{\gamma_d} g\left(\gamma_d\right)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\nu)\Bigg(\sum_{d\in\mathcal{F}(X)}\left[2\log\left(\frac{d}{2\pi}\right)+\frac{\Gamma'}{\Gamma}(6+i\nu)+\frac{\Gamma'}{\Gamma}(6-i\nu)\right]$$

$$+2\Bigg(-\frac{\zeta'}{\zeta}(1+2i\nu)+\frac{L'_\Delta}{L_\Delta}(sym^2,1+2i\nu)+B'_\Delta(i\nu;i\nu)$$

$$-\left(\frac{d}{2\pi}\right)^{-2it}\frac{\Gamma(6-i\nu)}{\Gamma(6+i\nu)}\frac{\zeta(1+2i\nu)L_\Delta(sym^2,1-2i\nu)}{L_\Delta(sym^2,1)}B_\Delta(-i\nu,i\nu)\Bigg)\Bigg)\,d\nu$$

$$+O(X^{1/2+\varepsilon}).$$

Matching terms (cont)

Easier piece to analyze:

$$-\frac{\zeta'}{\zeta}(1+2i\nu) + \frac{L'_\Delta}{L_\Delta}(sym^2, 1+2i\nu) = -\sum_p \sum_{k=1}^{\infty} \frac{(\alpha_p^{2k} + \overline{\alpha}_p^{2k}) \log p}{p^{k(1+2i\nu)}}.$$

Matching terms (cont)

Harder piece to analyze:

$$B_\Delta(\alpha;\gamma) = \prod_p \left(1 + \frac{p}{p+1}\left(\sum_{m=1}^\infty \frac{\tau^*(p^{2m})}{p^{m(1+2\alpha)}} - \frac{\tau^*(p)}{p^{1+\alpha+\gamma}}\sum_{m=0}^\infty \frac{\tau^*(p^{2m+1})}{p^{m(1+2\alpha)}} + \frac{1}{p^{1+2\gamma}}\sum_{m=0}^\infty \frac{\tau^*(p^{2m})}{p^{m(1+2\alpha)}}\right)\right)$$
$$\times \frac{\left(1 - \frac{\tau^*(p^2)}{p^{1+2\alpha}} + \frac{\tau^*(p^2)}{p^{2+4\alpha}} - \frac{1}{p^{3+6\alpha}}\right)\left(1 - \frac{1}{p^{1+2\gamma}}\right)}{\left(1 - \frac{\tau^*(p^2)}{p^{1+\alpha+\gamma}} + \frac{\tau^*(p^2)}{p^{2+2\alpha+2\gamma}} - \frac{1}{p^{3+3\alpha+3\gamma}}\right)\left(1 - \frac{1}{p^{1+\alpha+\gamma}}\right)}$$

Differentiating with respect to α and evaluating at $\alpha = \gamma = i\nu$, we have

$$B'_\Delta(i\nu; i\nu)$$
$$= \sum_p \log p \left(\frac{1}{p+1}\sum_{m=1}^\infty \frac{\alpha_p^{2m} + \overline{\alpha}_p^{2m}}{p^{m(1+2i\nu)}} - \sum_{m=1}^\infty \frac{\alpha_p^{2m} + \overline{\alpha}_p^{2m}}{p^{m(1+2i\nu)}} + \frac{\frac{\tau^*(p^2)}{p^{1+2i\nu}} - \frac{2\cdot\tau^*(p^2)}{p^{2+4i\nu}} + \frac{3}{p^{3+6i\nu}}}{1 - \frac{\tau^*(p^2)}{p^{1+2i\nu}} + \frac{\tau^*(p^2)}{p^{2+4i\nu}} - \frac{1}{p^{3+6i\nu}}} + \frac{1}{1 - p^{1+2i\nu}}\right)$$
$$= \sum_p \log p \left(\frac{1}{p+1}\sum_{m=1}^\infty \frac{\alpha_p^{2m} + \overline{\alpha}_p^{2m}}{p^{m(1+2i\nu)}} + 0\right)$$
$$= \sum_p \frac{\log p}{p+1}\sum_{m=1}^\infty \frac{\alpha_p^{2m} + \overline{\alpha}_p^{2m}}{p^{m(1+2i\nu)}}$$

Status

Need to show the following is small:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\nu)\left(\sum_{d\in\mathcal{F}(X)}\left(\frac{d}{2\pi}\right)^{-2it}\frac{\Gamma(6-i\nu)}{\Gamma(6+i\nu)}\frac{\zeta(1+2i\nu)L_\Delta(sym^2,1-2i\nu)}{L_\Delta(sym^2,1)}B_\Delta(-i\nu,i\nu)\right)d\nu$$

We have:

$$B_\Delta(-i\nu;i\nu)=\prod_p\left(1+\frac{p}{p+1}\left(\sum_{m=1}^{\infty}\frac{\tau^*(p^{2m})}{p^{m(1-2i\nu)}}-\frac{\tau^*(p)}{p}\sum_{m=0}^{\infty}\frac{\tau^*(p^{2m+1})}{p^{m(1-2i\nu)}}+\frac{1}{p^{1+2i\nu}}\sum_{m=0}^{\infty}\frac{\tau^*(p^{2m})}{p^{m(1-2i\nu)}}\right)\right)$$
$$\times\frac{\left(1-\frac{\tau^*(p^2)}{p^{1-2i\nu}}+\frac{\tau^*(p^2)}{p^{2-4i\nu}}-\frac{1}{p^{3-6i\nu}}\right)\left(1-\frac{1}{p^{1+2i\nu}}\right)}{\left(1-\frac{\tau^*(p^2)}{p}+\frac{\tau^*(p^2)}{p^2}-\frac{1}{p^3}\right)\left(1-\frac{1}{p}\right)}.$$

Conclusions

Conclusions

- Ratios Conjecture gives detailed predictions (up to $X^{1/2+\epsilon}$).
- Number Theory agrees with predictions for suitably restricted test functions.
- Numerics quite good.
- Similar results other families.
  ◇ All Dirichlet characters: SMALL '09.
  ◇ Quadratic twists of τ-function and a fixed elliptic curve: D. K. Huynh, S. J. Miller and R. Morrison.

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