Eigenvalue Statistics for Toeplitz and Circulant Ensembles

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Goals

- Review classical random matrix theory.
- See how the structure of the ensembles affects limiting behavior.
- Discuss the tools and techniques needed to prove the results.

Introduction

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \dots$.

Question: What rules govern the spacings between the t_i?

Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w $n^k \alpha$ mod 1.
- Spacings b/w Zeros of L-functions.

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Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.
2. Develop an explicit formula relating what we want to study to something we understand.
3. Use an averaging formula to analyze the quantities above.

Classical
Random Matrix Theory

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei (Uranium: 200$+$ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

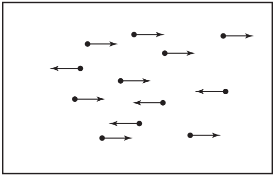
$$H\psi_n = E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Origins of Random Matrix Theory



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

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$$\mu_{A,N}(x) \quad = \quad \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

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$$\text{k}^{\text{th}} \text{ moment} = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.$$

Wigner's Semi-Circle Law

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) \;=\; \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) \;=\; \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

SKETCH OF PROOF: Correct Scale

$$\mathrm{Trace}(A^2) \;=\; \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\mathrm{Trace}(A^2) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \;\sim\; N^2$$

$$\sum_{i=1}^{N} \lambda_i(A)^2 \;\sim\; N^2$$

Gives $N\,\mathrm{Ave}(\lambda_i(A)^2) \;\sim\; N^2$ or $\mathrm{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

SKETCH OF PROOF: Averaging Formula

Recall k-th moment of $\mu_{A,N}(x)$ is $\mathrm{Trace}(A^k)/2^k N^{k/2+1}$.

Average k-th moment is

$$\int \cdots \int \frac{\mathrm{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \le j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of k-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).

SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

$$\frac{1}{2^2N^2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ji}^2\cdot p(a_{11})da_{11}\cdots p(a_{NN})da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty}a_{ij}^2 p(a_{ij})da_{ij}\quad\cdot\prod_{\substack{(k,l)\neq(i,j)\\ k<l}}\int_{a_{kl}=-\infty}^{\infty}p(a_{kl})da_{kl}\ =\ 1.$$

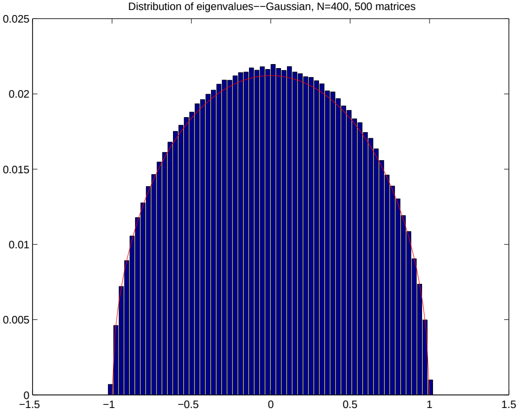
SKETCH OF PROOF: Averaging Formula for Higher Moments

Higher moments involve more advanced combinatorics (Catalan numbers).

$$\frac{1}{2^k N^{k/2+1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} \cdots a_{i_k i_1} \cdot \prod_{i \le j} p(a_{ij}) da_{ij}.$$

Main contribution when the $a_{i_\ell i_{\ell+1}}$'s matched in pairs, not all matchings contribute equally (if did would get a Gaussian and not a semi-circle; this is seen in Real Symmetric Palindromic Toeplitz matrices).

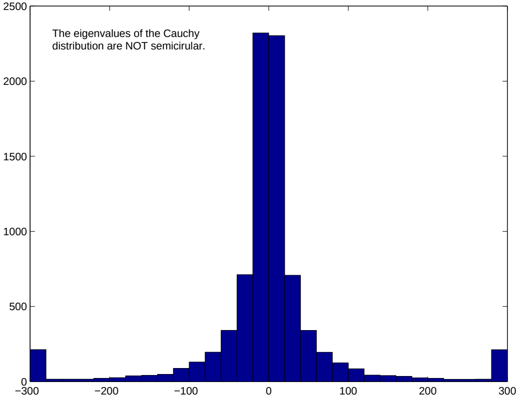
Numerical examples



500 Matrices: Gaussian 400×400

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Numerical examples



Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$

GOE Conjecture

GOE Conjecture:

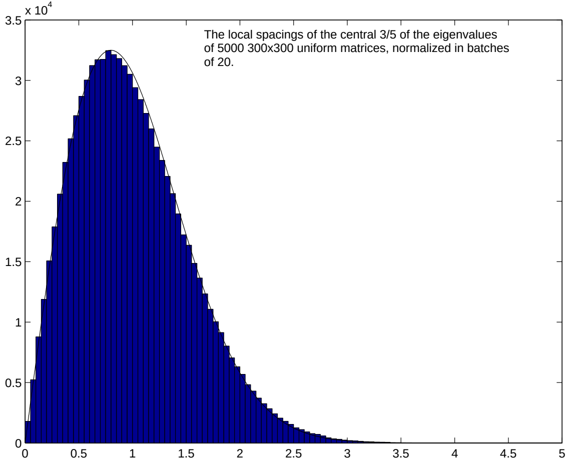
As $N \to \infty$, the probability density of the spacing b/w consecutive normalized eigenvalues approaches a limit independent of p.

Until recently only known if p is a Gaussian.

$$\mathrm{GOE}(x) \ \approx \ \frac{\pi}{2} x e^{-\pi x^2/4}.$$

Numerical Experiment: Uniform Distribution

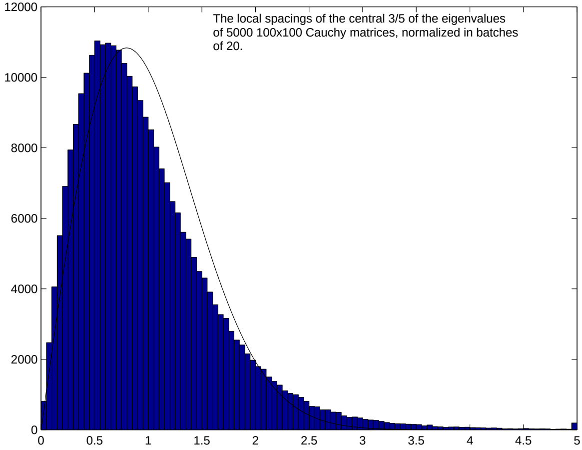
Let $p(x) = \frac{1}{2}$ for $|x| \leq 1$.



5000: 300×300 uniform on $[-1, 1]$

Cauchy Distribution

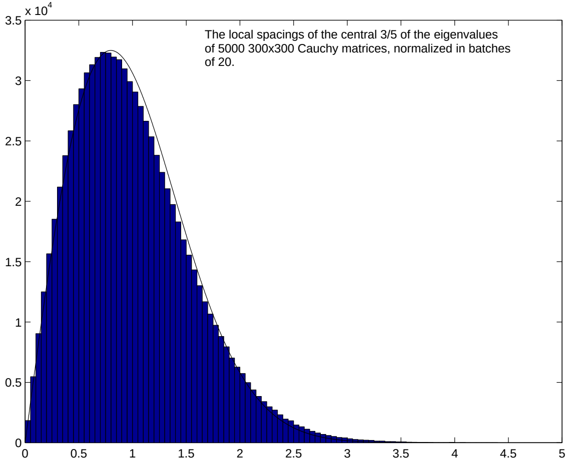
Let $p(x) = \frac{1}{\pi(1+x^2)}$.



5000: 100 $\times$ 100 Cauchy

Cauchy Distribution

Let $p(x) = \frac{1}{\pi(1+x^2)}$.



5000: 300×300 Cauchy

Fat-Thin
Families

Fat-Thin Families

Need a family FAT enough to do averaging and THIN enough so that everything isn't averaged out.

Real Symmetric Matrices have $\frac{N(N+1)}{2}$ independent entries.

Examples of Fat-Thin sub-families:

- Band Matrices
- Random Graphs
- Special Matrices (Toeplitz)

Band Matrices

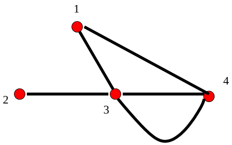
Example of a Band 1 Matrix:

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & \cdots & 0 \\ a_{12} & a_{22} & a_{23} & 0 & \cdots & 0 \\ 0 & a_{23} & a_{33} & a_{24} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & a_{N-1,N} \\ 0 & 0 & 0 & \cdot\cdot & a_{N-1,N} & a_{NN} \end{pmatrix}$$

For Band 0 (Diagonal Matrices):

- Density of Eigenvalues: $p(x)$
- Spacings b/w eigenvalues: Poissonian.

Random Graphs



Degree of a vertex $=$ number of edges leaving the vertex.
Adjacency matrix: a_{ij} $=$ number edges b/w Vertex i and Vertex j.

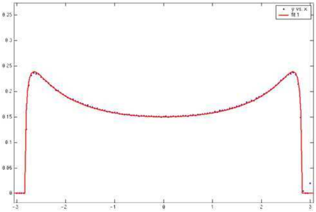
$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

These are Real Symmetric Matrices.

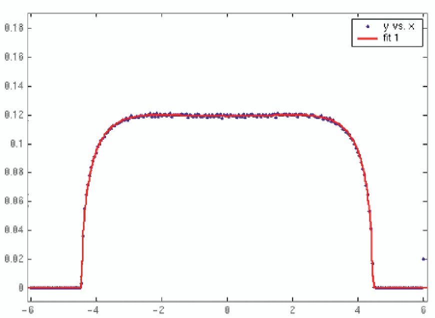
McKay's Law (Kesten Measure) with $d = 3$

Density of Eigenvalues for d-regular graphs

$$f(x) = \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2} & |x| \leq 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$



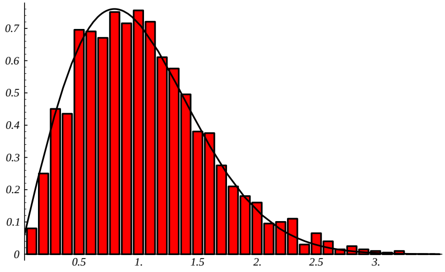
McKay's Law (Kesten Measure) with $d = 6$



Fat-Thin: fat enough to average, thin enough to get something different than semi-circle (though as $d \to \infty$ recover semi-circle).

3-Regular Graph with 2000 Vertices: Comparison with the GOE

Spacings between eigenvalues of 3-regular graphs and the GOE:



Real Symmetric Toeplitz Matrices

Chris Hammond and Steven J. Miller

Toeplitz Ensembles

Toeplitz matrix is of the form

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} \\ b_{-1} & b_0 & b_1 & \cdots & b_{N-2} \\ b_{-2} & b_{-1} & b_0 & \cdots & b_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{1-N} & b_{2-N} & b_{3-N} & \cdots & b_0 \end{pmatrix}$$

- Will consider Real Symmetric Toeplitz matrices.
- Main diagonal zero, $N-1$ independent parameters.
- Normalize Eigenvalues by $\sqrt{N}$.

Eigenvalue Density Measure

$$\mu_{A,N}(x)dx \ = \ \frac{1}{N}\sum_{i=1}^{N}\delta\left(x-\frac{\lambda_i(A)}{\sqrt{N}}\right)dx.$$

The k^{th} moment of $\mu_{A,N}(x)$ is

$$M_k(A,N) \ = \ \frac{1}{N^{\frac{k}{2}+1}}\sum_{i=1}^{N}\lambda_i^k(A) \ = \ \frac{\mathrm{Trace}(A^k)}{N^{\frac{k}{2}+1}}.$$

Let

$$M_k(N) \ = \ \lim_{N\to\infty} M_k(A,N).$$

Moments: $k = 2$ and k odd

Lemma: $M_2(N) \to 1$: As $a_{ij} = b_{|i-j|}$:

$$\begin{aligned} M_2(N) &= \frac{1}{N^2} \sum_{1 \le i_1, i_2 \le N} \mathbb{E}(b_{|i_1-i_2|} b_{|i_2-i_1|}) \\ &= \frac{1}{N^2} \sum_{1 \le i_1, i_2 \le N} \mathbb{E}(b^2_{|i_1-i_2|}). \end{aligned}$$

$N^2 - N$ times get 1, N times 0, thus $M_2(N) = 1 - \frac{1}{N}$. □

Lemma: $M_{2k+1}(N) \to 0$: Follows from trivial counting.

Even Moments

$$M_{2k}(N) = \frac{1}{N^{k+1}} \sum_{1 \le i_1, \cdots, i_{2k} \le N} \mathbb{E}(b_{|i_1-i_2|} b_{|i_2-i_3|} \cdots b_{|i_{2k}-i_1|}).$$

Main Term: b_j's matched in pairs, say

$$b_{|i_m-i_{m+1}|} = b_{|i_n-i_{n+1}|}, \quad x_m = |i_m - i_{m+1}| = |i_n - i_{n+1}|.$$

Two possibilities:

$$i_m - i_{m+1} = i_n - i_{n+1} \quad \text{or} \quad i_m - i_{m+1} = -(i_n - i_{n+1}).$$

$(2k-1)!!$ ways to pair, 2^k choices of sign.

Main Term: All Signs Negative (else lower order contribution)

$$M_{2k}(N) = \frac{1}{N^{k+1}} \sum_{1\le i_1,\cdots,i_{2k}\le N} \mathbb{E}(b_{|i_1-i_2|}b_{|i_2-i_3|}\cdots b_{|i_{2k}-i_1|}).$$

Let $x_1,\dots,x_k$ be the values of the $|i_j - i_{j+1}|$'s, $\epsilon_1,\dots,\epsilon_k$ the choices of sign. Define $\widetilde{x}_1 = i_1 - i_2$, $\widetilde{x}_2 = i_2 - i_3, \dots$.

$$\begin{aligned}
i_2 &= i_1 - \widetilde{x}_1 \\
i_3 &= i_1 - \widetilde{x}_1 - \widetilde{x}_2 \\
&\vdots \\
i_1 &= i_1 - \widetilde{x}_1 - \cdots - \widetilde{x}_{2k} \\
\widetilde{x}_1 + \cdots + \widetilde{x}_{2k} &= \sum_{j=1}^{k}(1+\epsilon_j)\eta_j x_j \ = \ 0, \quad \eta_j \ = \ \pm 1.
\end{aligned}$$

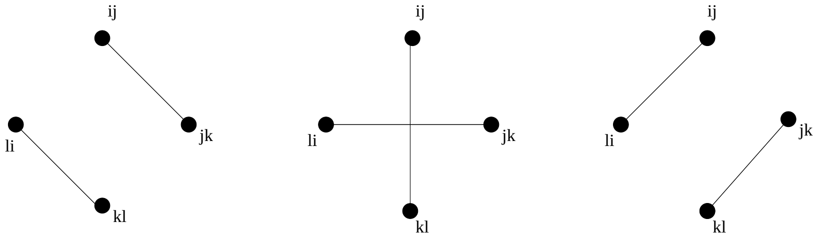
Even Moments: Summary

Main Term: paired, all signs negative.

$$M_{2k}(N) \ \leq \ (2k-1)!! + O_k\left(\frac{1}{N}\right).$$

Bounded by Gaussian.

The Fourth Moment



$$M_4(N) = \frac{1}{N^3} \sum_{1 \le i_1, i_2, i_3, i_4 \le N} \mathbb{E}(b_{|i_1 - i_2|} b_{|i_2 - i_3|} b_{|i_3 - i_4|} b_{|i_4 - i_1|})$$

Let $x_j = |i_j - i_{j+1}|$.

The Fourth Moment

Case One: $x_1 = x_2, x_3 = x_4$**:**

$$i_1 - i_2 = -(i_2 - i_3) \quad \text{and} \quad i_3 - i_4 = -(i_4 - i_1).$$

Implies

$$i_1 = i_3, \quad i_2 \text{ and } i_4 \text{ arbitrary.}$$

Left with $\mathbb{E}[b_{x_1}^2 b_{x_3}^2]$:

$$N^3 - N \text{ times get } 1, \; N \text{ times get } p_4 = \mathbb{E}[b_{x_1}^4].$$

Contributes 1 in the limit.

The Fourth Moment

$$M_4(N) = \frac{1}{N^3} \sum_{1 \le i_1, i_2, i_3, i_4 \le N} \mathbb{E}(b_{|i_1 - i_2|} b_{|i_2 - i_3|} b_{|i_3 - i_4|} b_{|i_4 - i_1|})$$

Case Two: Diophantine Obstruction: $x_1 = x_3$ and $x_2 = x_4$.

$$i_1 - i_2 \;=\; -(i_3 - i_4) \quad \text{and} \quad i_2 - i_3 \;=\; -(i_4 - i_1).$$

This yields

$$i_1 \;=\; i_2 + i_4 - i_3, \quad i_1, i_2, i_3, i_4 \in \{1, \dots, N\}.$$

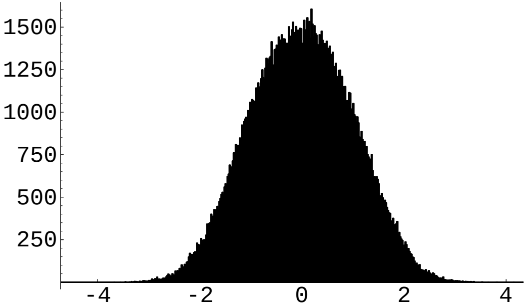
If $i_2, i_4 \geq \frac{2N}{3}$ and $i_3 < \frac{N}{3}$, $i_1 > N$: at most $(1 - \frac{1}{27})N^3$ valid choices.

The Fourth Moment

Theorem: Fourth Moment: Let p_4 be the fourth moment of p. Then

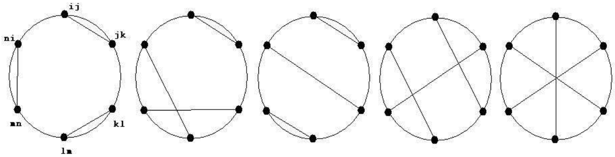
$$M_4(N) = 2\frac{2}{3} + O_{p_4}\left(\frac{1}{N}\right).$$

500 Toeplitz Matrices, 400×400.



Higher Moments: Brute Force Computations

For sixth moment, five configurations occurring (respectively) $2, 6, 3, 3$ and 1 times.



$M_6(N) = 11$ (Gaussian's is 15).
$M_8(N) = 64\frac{4}{15}$ (Gaussian's is 105).

Lemma: For $2k \geq 4$, $\lim_{N\to\infty} M_{2k}(N) < (2k-1)!!$.

Higher Moments: Unbounded support

Lemma: Moments' growth implies unbounded support.

Proof: Main idea:

$$\begin{aligned} i_2 &= i_1 - \widetilde{x}_1 \\ i_3 &= i_1 - \widetilde{x}_1 - \widetilde{x}_2 \\ &\vdots \\ i_{2k} &= i_1 - \widetilde{x}_1 - \cdots - \widetilde{x}_{2k}. \end{aligned}$$

Once specify i_1 and $\widetilde{x}_1$ through $\widetilde{x}_{2k}$, all indices fixed.
If matched in pairs and each $i_j \in \{1, \dots, N\}$, have a valid configuration, contributes $+1$.
Problem: a running sum $i_1 - \widetilde{x}_1 - \cdots - \widetilde{x}_m \notin \{1, \dots, N\}$.
Lots of freedom in locating positive and negative signs, use CLT to show "most" configurations are valid.

Main Result

Types of Convergence: Define the random variable $X_{m;N}$ on $\Omega_{\mathbb{N}}$ by

$$X_{m;N}(A) = \int_{-\infty}^{\infty} x^m dF^{A_N/\sqrt{N}}(x);$$

note this is the m^{th} moment of the measure μ_{A_N}.

1. Almost sure convergence: For each m, $X_{m;N} \to X_m$ almost surely if $\mathbb{P}_{\mathbb{N}}\left(\{A \in \Omega_{\mathbb{N}} : X_{m;N}(A) \to X_m(A) \text{ as } N \to \infty\}\right) = 1$;
2. In probability: For each m, $X_{m;N} \to X_m$ in probability if for all $\epsilon > 0$, $\lim_{N\to\infty} \mathbb{P}_{\mathbb{N}}(|X_{m;N}(A) - X_m(A)| > \epsilon) = 0$;
3. Weak convergence: For each m, $X_{m;N} \to X_m$ weakly if

$$\mathbb{P}_{\mathbb{N}}(X_{m;N}(A) \leq x) \to \mathbb{P}(X_m(A) \leq x)$$

as $N \to \infty$ for all x at which $F_{X_m}(x) = \mathbb{P}(X_m(A) \leq x)$ is continuous.

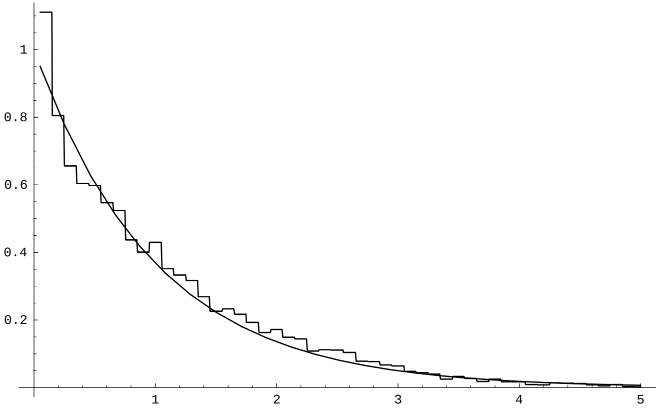
Main Result

Alternate notations are to say *with probability 1* for almost sure convergence and *in distribution* for weak convergence; both almost sure convergence and convergence in probability imply weak convergence. For our purposes we take X_m as the random variable which is identically M_m (thus $X_m(A) = M_m$ for all $A \in \Omega_{\mathbb{N}}$).

Theorem: HM '05

For real symmetric Toeplitz matrices, the limiting spectral measure converges in probability to a unique measure of unbounded support which is not the Gaussian. If p is even have strong convergence).

Poissonian Behavior?



Not rescaled. Looking at middle 11 spacings, 1000 Toeplitz matrices (1000×1000), entries iidrv from the standard normal.

Real Symmetric Palindromic Toeplitz Matrices

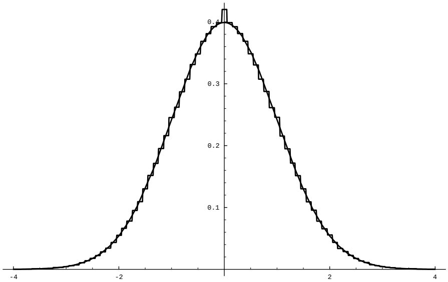
Adam Massey, Steven J. Miller, Jon Sinsheimer

Real Symmetric Palindromic Toeplitz matrices

$$\begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \cdots & b_3 & b_2 & b_1 & b_0 \\ b_1 & b_0 & b_1 & b_2 & \cdots & b_4 & b_3 & b_2 & b_1 \\ b_2 & b_1 & b_0 & b_1 & \cdots & b_5 & b_4 & b_3 & b_2 \\ b_3 & b_2 & b_1 & b_0 & \cdots & b_6 & b_5 & b_4 & b_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ b_3 & b_4 & b_5 & b_6 & \cdots & b_0 & b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 & b_5 & \cdots & b_1 & b_0 & b_1 & b_2 \\ b_1 & b_2 & b_3 & b_4 & \cdots & b_2 & b_1 & b_0 & b_1 \\ b_0 & b_1 & b_2 & b_3 & \cdots & b_3 & b_2 & b_1 & b_0 \end{pmatrix}$$

- Extra symmetry fixes Diophantine Obstructions.
- Always have eigenvalue at 0.

Real Symmetric Palindromic Toeplitz (cont)



500 Real Symmetric Palindromic Toeplitz, 1000×1000.

Note the bump at the zeroth bin is due to the forced eigenvalues at 0.

Effects of Palindromicity on Matchings

$a_{i_m i_{m+1}}$ paired with $a_{i_n i_{n+1}}$ implies one of the following hold:

$$\begin{aligned} i_{m+1} - i_m &= \pm(i_{n+1} - i_n) \\ i_{m+1} - i_m &= \pm(i_{n+1} - i_n) + (N-1) \\ i_{m+1} - i_m &= \pm(i_{n+1} - i_n) - (N-1). \end{aligned}$$

Concisely: There is a $C \in \{0, \pm(N-1)\}$ such that

$$i_{m+1} - i_m = \pm(i_{n+1} - i_n) + C.$$

Fourth Moment

Highlights the effect of palindromicity.

Still matched in pairs, but more diagonals now lead to valid matchings.

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Non-adjacent case was $x_1 = x_3$ and $x_2 = x_4$:

$$i_1 - i_2 \;=\; -(i_3 - i_4) \quad \text{and} \quad i_2 - i_3 \;=\; -(i_4 - i_1).$$

This yields

$$i_1 \;=\; i_2 + i_4 - i_3, \quad i_1, i_2, i_3, i_4 \in \{1, \dots, N\}.$$

Fourth Moment

Highlights the effect of palindromicity.

Still matched in pairs, but more diagonals now lead to valid matchings.

Non-adjacent case now $x_1 = x_3$ and $x_2 = x_4$:

$$j - i \;=\; -(l - k) + C_1 \qquad\qquad k - j \;=\; -(i - l) + C_2,$$

or equivalently

$$j \;=\; i + k - l + C_1 \;=\; i + k - l - C_2.$$

We see that $C_1 = -C_2$, or $C_1 + C_2 = 0$.

Results

Theorem: MMS '07

For real symmetric palindromic matrices, converge in probability to the Gaussian (if p is even have strong convergence).

Results

Theorem: MMS '07

Let $X_0, \dots, X_{N-1}$ be iidrv (with $X_j = X_{N-j}$) from a distribution p with mean 0, variance 1, and finite higher moments. For $\omega = (x_0, x_1, \dots)$ set $X_\ell(\omega) = x_\ell$, and

$$S_N^{(k)}(\omega) \ = \ \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} X_\ell(\omega) \cos(2\pi k \ell / N).$$

Then as $n \to \infty$

$$\text{Prob}\left(\left\{\omega \in \Omega : \sup_{x \in \mathbb{R}} \left| \frac{1}{N} \sum_{k=0}^{N-1} I_{S_N^{(k)}(\omega) \le x} - \Phi(x) \right| \to 0\right\}\right) \ = \ 1;$$

I the indicator fn, Φ CDF of standard normal.

Summary

Ensemble	order D.F.	Density	Spacings
Real Symm	N^2	Semi-Circle	GOE
Diagonal	N	$p(x)$	Poisson
d-Regular	dN	Kesten	GOE
Toeplitz	N	Toeplitz	Poisson
Palindromic Toeplitz	N	Gaussian	

Red is conjectured
Blue is recent

Real Symmetric Highly Palindromic Toeplitz Matrices

Steven Jackson, Steven J. Miller, Vincent Pham

Notation: Real Symmetric Highly Palindromic Toeplitz matrices

For fixed n, we consider $N \times N$ real symmetric Toeplitz matrices in which the first row is 2^n copies of a palindrome, entries are iidrv from a p with mean 0, variance 1 and finite higher moments.

For instance, a doubly palindromic Toeplitz matrix is of the form:

$$A_N = \begin{pmatrix} b_0 & b_1 & \cdots & b_1 & b_0 & b_0 & b_1 & \cdots & b_1 & b_0 \\ b_1 & b_0 & \cdots & b_2 & b_1 & b_0 & b_0 & \cdots & b_2 & b_1 \\ b_2 & b_1 & \cdots & b_3 & b_2 & b_1 & b_0 & \cdots & b_3 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_2 & b_3 & \cdots & b_0 & b_1 & b_2 & b_3 & \cdots & b_1 & b_2 \\ b_1 & b_2 & \cdots & b_0 & b_0 & b_1 & b_2 & \cdots & b_0 & b_1 \\ b_0 & b_1 & \cdots & b_1 & b_0 & b_0 & b_1 & \cdots & b_1 & b_0 \end{pmatrix}.$$

Main Results

Theorem: JMP '09

Let n be a fixed positive integer, N a multiple of 2^n, consider the ensemble of real symmetric $N \times N$ palindromic Toeplitz matrices whose first row is 2^n copies of a fixed palindrome (independent entries iidrv from p with mean 0, variance 1 and finite higher moments).

1. As $N \to \infty$ the measures μ_{n,A_N} converge in probability to a limiting spectral measure which is even and has unbounded support.
2. If p is even, then converges almost surely.
3. The limiting measure has fatter tails than the Gaussian (or any previously seen distribution).

Key Lemmas

Much of analysis similar to previous ensembles (though combinatorics more involved).

For the fourth moment: both the adjacent and non-adjacent matchings contribute the same.

Lemma: As $N \to \infty$ the fourth moment tends to

$$M_{4,n} \;=\; 2^{n+1} + 2^{-n}.$$

Note: Number of palindromes is 2^n; thus smallest is $2^0 = 1$ (and do recover 3 for palindromic Toeplitz).

Conjectures

Conjecture

In the limit, all matchings contribute equally.

Very hard to test; numerics hard to analyze.

To avoid simulating ever-larger matrices, noticed Diophantine analysis suggests average $2m^{\text{th}}$ moment of $N \times N$ matrices should satisfy

$$M_{2m,n;N} \;=\; M_{2m,n} + \frac{C_{1,n}}{N} + \frac{C_{2,n}}{N^2} + \cdots + \frac{C_{m,n}}{N^m}.$$

Instead of simulating prohibitively large matrices, simulate large numbers of several sizes of smaller matrices, do a least squares analysis to estimate $M_{2m,n}$.

Conjectures

Table: Conjectured and observed moments for 1000 real symmetric doubly palindromic 2048 $\times$ 2048 Toeplitz matrices. The conjectured values come from assuming Conjecture.

Moment	Conjectured	Observed	Observed/Predicted
2	1.000	1.001	1.001
4	4.500	4.521	1.005
6	37.500	37.887	1.010
8	433.125	468.53	1.082
10	6260.63	107717.3	17.206

Conjectures

Table: Observed moments for doubly palindromic Toeplitz matrices. Conjectured values from assuming Conjecture.

N	#sims	2nd	4th	6th	8th	10th
8	1,000,000	1.000	8.583	150.246	3984.36	141270.00
12	1,000,000	1.000	7.178	110.847	2709.61	90816.60
16	1,000,000	1.001	6.529	93.311	2195.78	73780.00
20	1,000,000	1.001	6.090	80.892	1790.39	57062.50
24	1,000,000	1.000	5.818	73.741	1577.42	49221.50
28	1,000,000	1.000	5.621	68.040	1396.50	42619.90
64	250,000	1.001	4.992	50.719	858.58	22012.90
68	250,000	1.000	4.955	49.813	831.66	20949.60
72	250,000	1.000	4.933	49.168	811.50	20221.20
76	250,000	1.000	4.903	48.474	794.10	19924.10
80	250,000	1.000	4.888	47.951	773.31	18817.00
84	250,000	1.001	4.876	47.615	764.84	18548.00
128	125,000	1.000	4.745	44.155	659.00	14570.60
132	125,000	1.000	4.739	43.901	651.18	14325.30
136	125,000	0.999	4.718	43.456	637.70	13788.10
140	125,000	1.000	4.718	43.320	638.74	14440.40
144	125,000	1.001	4.727	43.674	647.05	14221.80
148	125,000	1.000	4.716	43.172	628.02	13648.10
Conjectured		1.000	4.500	37.500	433.125	6260.63
Best Fit $M_{2m,2}$		1.000	4.496	38.186	490.334	6120.94

Period m Circulant Matrices

Gene Kopp, Murat Koloğlu and Steven J. Miller

Study circulant matrices periodic with period m on diagonals.

6-by-6 real symmetric period 2-circulant matrix:

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_2 & d_1 \\ c_1 & d_0 & d_1 & d_2 & c_3 & d_2 \\ c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\ c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\ c_2 & c_3 & c_2 & d_1 & c_0 & c_1 \\ d_1 & d_2 & c_3 & d_2 & c_1 & d_0 \end{pmatrix}.$$

Look at the *expected value* for the moments:

$$\begin{aligned} M_n(N) &:= \mathbb{E}(M_n(A,N)) \\ &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{1\le i_1,\dots,i_n\le N} \mathbb{E}(a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ni_1}). \end{aligned}$$

Matchings

Rewrite:

$$M_n(N) = \frac{1}{N^{\frac{n}{2}+1}} \sum_{\sim} \eta(\sim) m_{d_1(\sim)} \cdots m_{d_l(\sim)}.$$

where the sum is over equivalence relations on $\{(1,2),(2,3),...,(n,1)\}$. The $d_j(\sim)$ denote the sizes of the equivalence classes, and the m_d the moments of p. Finally, the coefficient $\eta(\sim)$ is the number of solutions to the system of Diophantine equations:
Whenever $(s, s+1) \sim (t, t+1)$,

- $i_{s+1} - i_s \equiv i_{t+1} - i_t \pmod{N}$ and $i_s \equiv i_t \pmod{m}$, or
- $i_{s+1} - i_s \equiv -(i_{t+1} - i_t) \pmod{N}$ and $i_s \equiv i_{t+1} \pmod{m}$.

- $i_{s+1} - i_s \equiv i_{t+1} - i_t \pmod{N}$ and $i_s \equiv i_t \pmod{m}$, or
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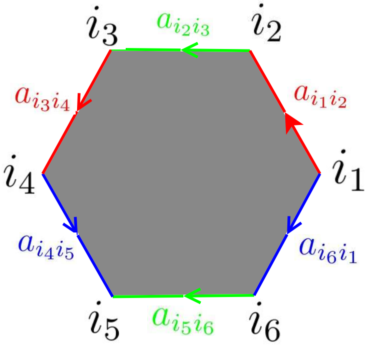
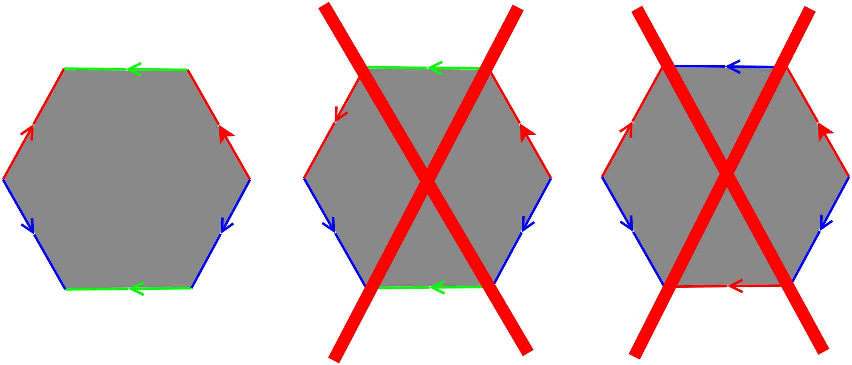


Figure: Red edges same orientation and blue, green opposite.

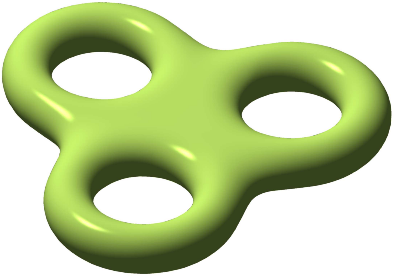
Contributing Terms

As $N \to \infty$, the only terms that contribute to this sum are those in which the entries are matched in pairs and with opposite orientation.



Algebraic Topology

Think of pairings as topological identifications, the contributing ones give rise to orientable surfaces.



Contribution from such a pairing is m^{-2g}, where g is the genus (number of holes) of the surface. Proof: combinatorial argument involving Euler characteristic.

Computing the Even Moments

Theorem: Even Moment Formula

$$M_{2k} = \sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) m^{-2g} + O_k\left(\frac{1}{N}\right),$$

with $\varepsilon_g(k)$ the number of pairings of the edges of a $(2k)$-gon giving rise to a genus g surface.

J. Harer and D. Zagier (1986) gave generating functions for the $\varepsilon_g(k)$.

Harer and Zagier

$$\sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) r^{k+1-2g} \;=\; (2k-1)!!\, c(k,r)$$

where

$$1 + 2\sum_{k=0}^{\infty} c(k,r) x^{k+1} \;=\; \left(\frac{1+x}{1-x}\right)^r.$$

Thus, we write

$$M_{2k} \;=\; m^{-(k+1)} (2k-1)!!\, c(k,m).$$

A multiplicative convolution and Cauchy's residue formula yields the *characteristic function* of the distribution (inverse Fourier transform of the density).

$$\begin{aligned}
\phi(t) &= \sum_{k=0}^{\infty} \frac{(it)^{2k} M_{2k}}{(2k)!} \\
&= \frac{1}{2\pi i m} \oint_{|z|=2} \frac{1}{2z^{-1}} \left(\left(\frac{1+z^{-1}}{1-z^{-1}} \right)^m - 1 \right) e^{-t^2 z/2m} \frac{dz}{z} \\
&= \frac{1}{m} e^{\frac{-t^2}{2m}} \sum_{l=1}^{m} \binom{m}{l} \frac{1}{(l-1)!} \left(\frac{-t^2}{m} \right)^{l-1}
\end{aligned}$$

Results

Fourier transform and algebra yields

Theorem: Kopp, Koloğlu and M–

The limiting spectral density function $f_m(x)$ of the real symmetric m-circulant ensemble is given by the formula

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m} \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s (mx^2)^r.$$

As $m \to \infty$, the limiting spectral densities approach the semicircle distribution.

Results (continued)

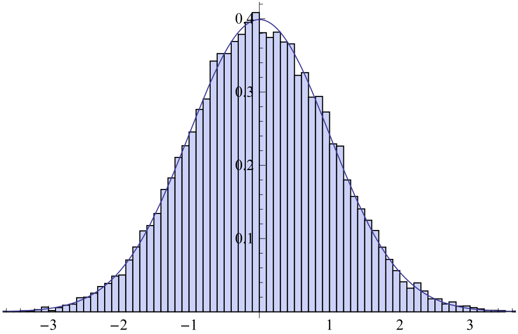


Figure: Plot for f_1 and histogram of eigenvalues of 100 circulant matrices of size 400×400.

Results (continued)

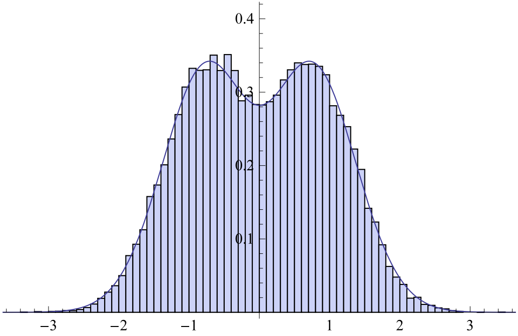


Figure: Plot for f_2 and histogram of eigenvalues of 100 2-circulant matrices of size 400×400.

Results (continued)

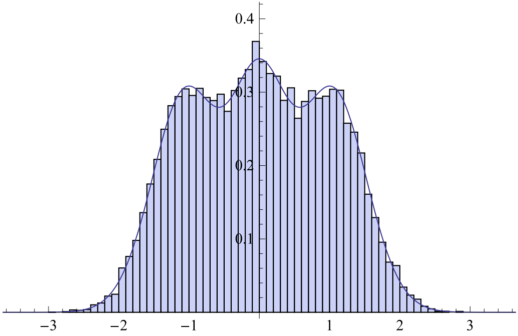


Figure: Plot for f_3 and histogram of eigenvalues of 100 3-circulant matrices of size 402×402.

Results (continued)

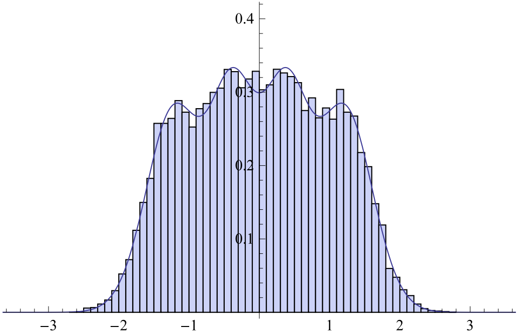


Figure: Plot for f_4 and histogram of eigenvalues of 100 4-circulant matrices of size 400×400.

Results (continued)

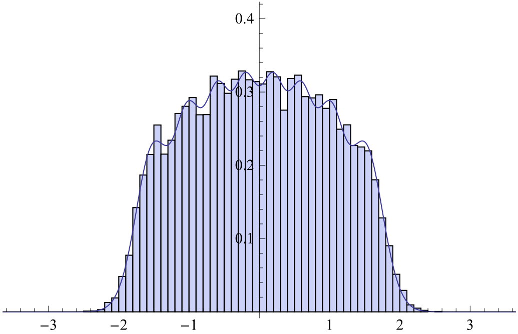


Figure: Plot for f_8 and histogram of eigenvalues of 100 8-circulant matrices of size 400×400.

Results (continued)

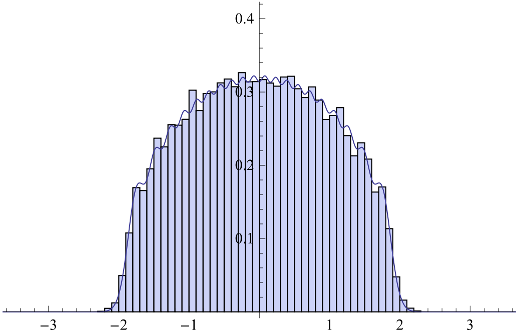


Figure: Plot for f_{20} and histogram of eigenvalues of 100 20-circulant matrices of size 400×400.

Results (continued)

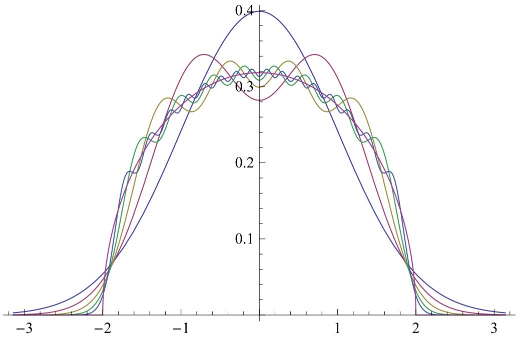


Figure: Plot of convergence to the semi-circle.

Real Symmetric Toeplitz Matrices

Olivia Beckwith, Steven J. Miller and Karen Shen

New Ensemble: Signed Toeplitz and Palindromic Toeplitz Matrices

For each entry, assign a randomly chosen $\epsilon_{ij} = \{1, -1\}$ such that $\epsilon_{ij} = \epsilon_{ji}$ with $p = \mathbb{P}(\epsilon_{ij} = 1)$.

Varying p allows us to *continuously* interpolate between:

- Real Symmetric at $p = \frac{1}{2}$ (less structured)
- Unsigned Toeplitz/Palindromic Toeplitz at $p = 1$ (more structured)

What is the eigenvalue distribution of these signed ensembles?

Weighted Contributions

Theorem:

Each configuration weighted by $(2p-1)^{2m}$, where $2m$ is the number of points on the circle whose edge crosses another edge.

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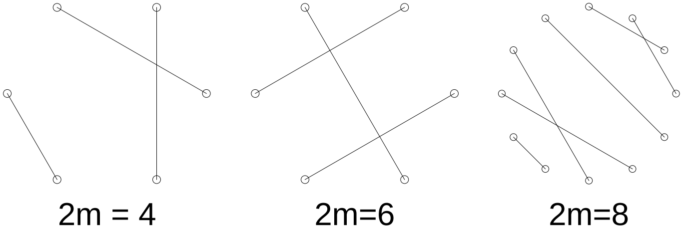
Example:



Proof of Weighted Contributions Theorem

For ϵ_{ij} to be matched with ϵ_{kl} (we know that $\epsilon_{ij} = \epsilon_{kl}$), it must be true that either $i = k$ and $j = l$ or $i = l$ and $j = k$.

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If ϵ_{ij} is matched with some ϵ_{kl}, then $\mathbb{E}\left(\epsilon_{ij}\epsilon_{kl}\right) = 1$.

If ϵ_{ij} is not matched with any ϵ_{kl}, then $\mathbb{E}\left(\epsilon_{ij}\right) = (2p - 1)$.

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If ϵ_{ij} is not matched with any ϵ_{kl}, then $\mathbb{E}\left(\epsilon_{ij}\right) = (2p - 1)$.

Want to prove that two ϵ's are matched if and only if their b's are not in a crossing.

Proof of Weighted Contributions Theorem

A non-crossing pair of b's must have matched ϵs:

Proof of Weighted Contributions Theorem

A non-crossing pair of *b*'s must have matched ϵs:

Assume $b_{|i_r - i_{r+1}|}$ and $b_{\left|i_p - i_{p+1}\right|}$ are a non-crossing pair.

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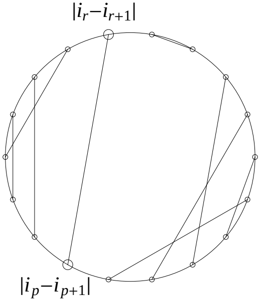

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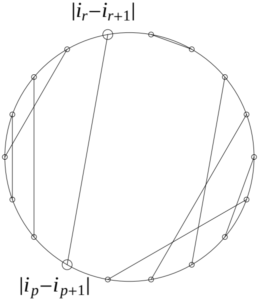
$\sum_{k=r}^{p} (i_k - i_{k+1}) = 0$



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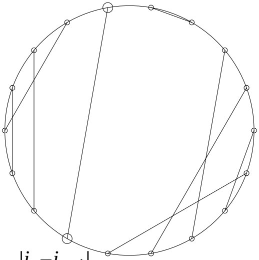

$$\sum_{k=r}^{p} (i_k - i_{k+1}) = 0$$
$$= i_r - i_{r+1} + i_{r+1} \cdots + i_p - i_{p+1} = i_r - i_{p+1}$$

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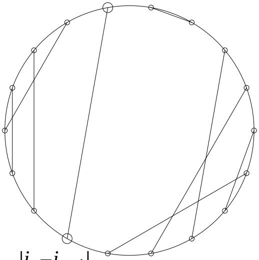

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This implies that $i_r = i_{p+1}$.

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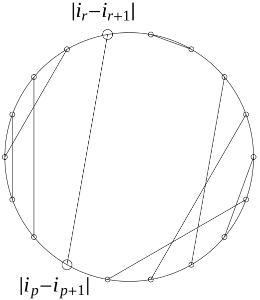

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Similarly, $i_{r+1} = i_p$

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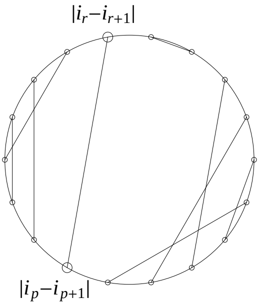

$$\sum_{k=r}^{p} (i_k - i_{k+1}) = 0$$
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This implies that $i_r = i_{p+1}$.
Similarly, $i_{r+1} = i_p$

Thus, $\epsilon_{i_r i_{r+1}} = \epsilon_{i_p i_{p+1}}$.

Proof of Weighted Contributions Theorem

A matched pair of ϵs must not be in a crossing:

Suppose $\epsilon_{i_a i_{a+1}} = \epsilon_{i_b i_{b+1}}$, with $a < b$.

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$$\begin{aligned} \sum_{k=a}^{b} (i_k - i_{k+1}) &= i_a - i_{b+1} = 0 \\ &= \sum_{k=b}^{d} \delta_k \left| i_k - i_{k+1} \right| \end{aligned}$$

where $\delta_k = 0$ if and only if the vertex k is paired with is between a and b.

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where $\delta_k = 0$ if and only if the vertex k is paired with is between a and b.

Need N^{k+1} degrees of freedom, so $\delta_k = 0$ for all k.

Proof of Weighted Contributions Theorem

A matched pair of ϵs must not be in a crossing:

Suppose $\epsilon_{i_a i_{a+1}} = \epsilon_{i_b i_{b+1}}$, with $a < b$.

$$\sum_{k=a}^{b} (i_k - i_{k+1}) = i_a - i_{b+1} = 0$$
$$= \sum_{k=b}^{d} \delta_k |i_k - i_{k+1}|$$

where $\delta_k = 0$ if and only if the vertex k is paired with is between a and b.

Need N^{k+1} degrees of freedom, so $\delta_k = 0$ for all k.
Thus, $\epsilon_{i_a i_{a+1}}$ and $\epsilon_{i_b i_{b+1}}$ are not in a crossing.

Counting Crossing Configurations

Problem: Out of the $(2k-1)!!$ ways to pair $2k$ vertices, how many will have $2m$ vertices crossing ($Cross_{2k,2m}$)?

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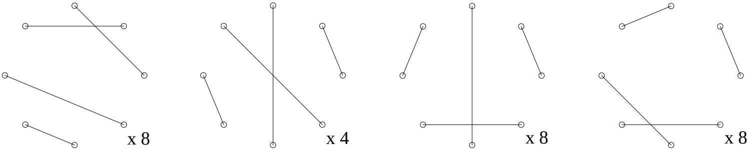
Example: $Cross_{8,4} = 28$



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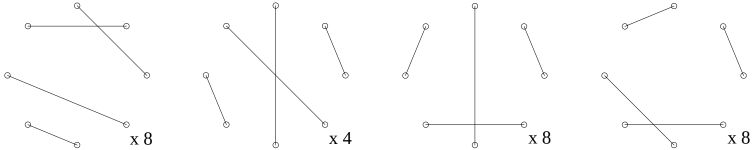

Fact:

$Cross_{2k,0} = C_k$, the k^{th} Catalan number.

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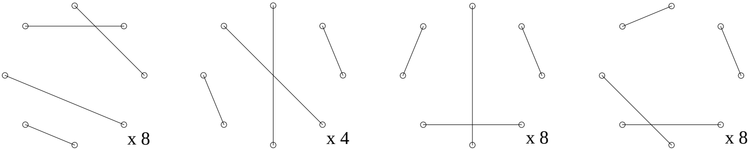


Fact:

$Cross_{2k,0} = C_k$, the k^{th} Catalan number.

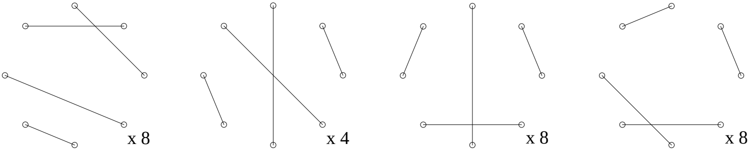
What about for higher m?

Non-Crossing Regions

Theorem:

Suppose $2m$ vertices are already paired in some configuration. The number of ways to pair and place the remaining $2k - 2m$ vertices such that none of them are involved in a crossing is $\binom{2k}{k-m}$.

Example: There are $\binom{8}{2} = 28$ pairings with 4 vertices arranged in a crossing.



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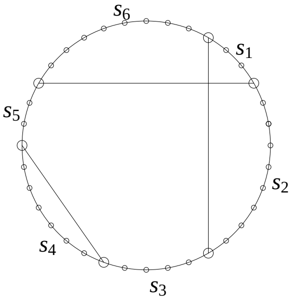

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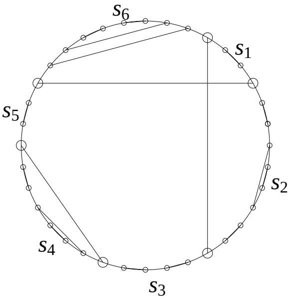

Counting Crossing Configurations

To calculate $Cross_{2k,2m}$, we write it as the following sum:

$$Cross_{2k,2m} = \sum_{p=1}^{\lfloor \frac{m}{4} \rfloor} P_{2k,2m,p}.$$

where $P_{2k,2m,p}$ is the number of configurations of $2k$ vertices with $2m$ vertices crossing in p partitions.

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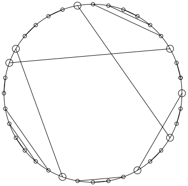
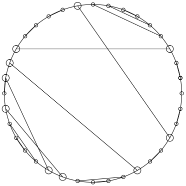


p = 1



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We then apply our theorem to get formulas for $P_{2k,2m,p}$. For example:

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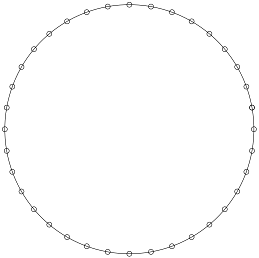

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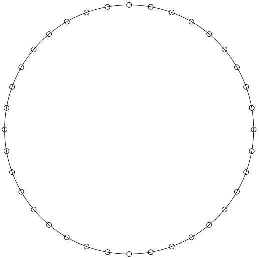

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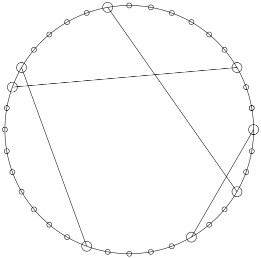

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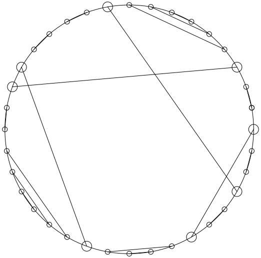

Counting Crossing Configurations

For:

$2k \setminus 2m$	0	4	6	8	10	Total
2						1
4						3
6						15
8						105
10						945
$\vdots$						

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For:

$2k \setminus 2m$	0	4	6	8	10	Total
2	1					1
4	2					3
6	5					15
8	14					105
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  - A way to count the number of configurations with $2m$ vertices crossing for all m
- The expected number of vertices involved in a crossing is

$$\frac{2k}{2k-1}\left(2k-2-\frac{{}_2F_1(1,3/2,5/2-k;-1)}{2k-3}-(2k-1)\,{}_2F_1(1,1/2+k,3/2;-1)\right),$$

  which is $2k-2-\frac{2}{k}+O\left(\frac{1}{k^2}\right)$ as $k\to\infty$.
- The variance tends to 4 as $k\to\infty$.