A Ramsey Theoretic Approach to Function Fields and Quaternions

Megumi Asada, Williams College
maa2@williams.edu

Sarah Manski, Kalamazoo College
sarah.manski12@kzoo.edu

http://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/UConn_FiniteFields_Quaternions.pdf

University of Connecticut
July 28, 2015

Classical Ramsey Theory

Ramsey Theory

Ramsey Theory is concerned with seeing how large a collection of objects can be while avoiding a particular substructure.

Classical Ramsey Theory

Ramsey Theory

Ramsey Theory is concerned with seeing how large a collection of objects can be while avoiding a particular substructure.

Friends and Strangers Problem

What is the smallest group of people needed to guarantee k friends or k strangers?

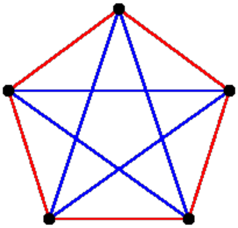
Classical Ramsey Theory

Ramsey Theory

Ramsey Theory is concerned with seeing how large a collection of objects can be while avoiding a particular substructure.

Friends and Strangers Problem

What is the smallest group of people needed to guarantee k friends or k strangers?



Previous Work

Rankin (1961)

Integers avoiding geometric progressions: $a, \ ab, \ ab^2$ with $a, b \in \mathbb{Z}$

Previous Work

Rankin (1961)

Integers avoiding geometric progressions: $a, \ ab, \ ab^2$ with $a, b \in \mathbb{Z}$

3-Term Arithmetic-free Sequence in $\mathbb{Z}$:
$A_3^*(\mathbb{Z}) \ = \ \{0, \ 1, \ 3, \ 4, \ 9, \ 10, \ 12, \ 13, \ \dots \}$

Only include integers with prime exponents in $A_3^*(\mathbb{Z})$

Previous Work

Rankin (1961)

Integers avoiding geometric progressions: $a, \ ab, \ ab^2$ with $a, b \in \mathbb{Z}$

3-Term Arithmetic-free Sequence in $\mathbb{Z}$:
$A_3^*(\mathbb{Z}) \ = \ \{0, \ 1, \ 3, \ 4, \ 9, \ 10, \ 12, \ 13, \ \dots \}$

Only include integers with prime exponents in $A_3^*(\mathbb{Z})$

Greedy Set asymptotic density ≈ 0.71974

Previous Work

SMALL '14: Generalization to Number Fields

The density of the greedy set of ideals which avoid progressions with rational integer ratios is ≈ 0.939735.

Preliminaries

Functon Field

We view $\mathbb{F}_q[x]$, *with* $q = p^n$, *as the ring of all polynomials with coefficients in the finite field* $\mathbb{F}_q$.

Preliminaries

Functon Field

We view $\mathbb{F}_q[x]$, with $q = p^n$, as the ring of all polynomials with coefficients in the finite field $\mathbb{F}_q$.

Goal

Construct a Greedy Set of polynomials in $\mathbb{F}_q[x]$ free of geometric progressions.

The Greedy Set

- Rewrite any $f(x)$ as $f(x) = uP_1^{\alpha_1} \cdots P_k^{\alpha_k}$ where u is a unit, and each P_i is a monic irreducible polynomial.
- Exclude $f(x)$ with $\alpha_i \notin A_3^*(\mathbb{Z}) = \{0, 1, 3, 4, 9, 10, 12, 13, \ldots\}$.

The Greedy Set

- Rewrite any $f(x)$ as $f(x) = uP_1^{\alpha_1} \cdots P_k^{\alpha_k}$ where u is a unit, and each P_i is a monic irreducible polynomial.
- Exclude $f(x)$ with $\alpha_i \notin A_3^*(\mathbb{Z}) = \{0, 1, 3, 4, 9, 10, 12, 13, \dots\}$.

Greedy Set in $\mathbb{F}_q[x]$

The Greedy Set is exactly the set of all $f(x) \in \mathbb{F}_q[x]$ only with prime exponents in $A_3^*(\mathbb{Z})$

Asymptotic Density

The *asymptotic density of the greedy set* $G_{3,q}^* \subseteq \mathbb{F}_q[x]$ can be expressed as

$$d(G_3^*) \;=\; \left(1 - \frac{1}{q}\right) \prod_{i=1}^{\infty} \prod_{n=1}^{\infty} \left(1 + q^{-n3^i}\right)^{m(n)},$$

where $m(n) \;=\; \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) q^d$ gives the number of monic irreducibles over $\mathbb{F}_q[x]$.

Asymptotic Density

The *asymptotic density of the greedy set* $G^*_{3,q} \subseteq \mathbb{F}_q[x]$ can be expressed as

$$d(G^*_3) = \left(1 - \frac{1}{q}\right) \prod_{i=1}^{\infty} \prod_{n=1}^{\infty} \left(1 + q^{-n3^i}\right)^{m(n)},$$

where $m(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) q^d$ gives the number of monic irreducibles over $\mathbb{F}_q[x]$.
Becomes a lower bound when truncated.

Lower Bound

Table: Lower Bound for Density of $G_3^*(\mathbb{F}_q[x])$.

q	$d(G_3^*)$ for $\mathbb{F}_q[x]$
2	.648361
3	.747027
4	.799231
5	.833069
7	.874948
8	.888862

q	$d(G_3^*)$ for $\mathbb{F}_q[x]$
9	.899985
25	.961538
27	.964286
49	.980000
125	.992063
343	.997093

Bounds on Upper Densities

Table: New upper bounds (q-smooth) compared to the old upper bounds, as well as the lower bounds for the supremum of upper densities.

q	New Bound (q-smooth)	Old Bound	Lower Bound
2	.846435547	.857142857	.845397956
3	.921933009	.923076923	.921857532
4	.967684196	.96774193	.967680495
5	.967684196	.967741935	.967680495
7	.982448450	.982456140	.982447814

Types of Quaternions

Definition

Quaternions constitute the algebra over the reals generated by units i, j, and k such that

$$i^2 = j^2 = k^2 = ijk = -1.$$

Quaternions can be written as $a + bi + cj + dk$ for $a, b, c, d \in \mathbb{R}$.

Types of Quaternions

Definition

Quaternions constitute the algebra over the reals generated by units i, j, and k such that

$$i^2 = j^2 = k^2 = ijk = -1.$$

Quaternions can be written as $a + bi + cj + dk$ for $a, b, c, d \in \mathbb{R}$.

Definition

We say that $a + bi + cj + dk$ is in the Hurwitz Order, $\mathcal{H}$ if a, b, c, d are all integers or halves of odd integers.

Types of Quaternions

Definition

Quaternions constitute the algebra over the reals generated by units i, j, and k such that

$$i^2 = j^2 = k^2 = ijk = -1.$$

Quaternions can be written as $a + bi + cj + dk$ for $a, b, c, d \in \mathbb{R}$.

Definition

We say that $a + bi + cj + dk$ is in the Hurwitz Order, $\mathcal{H}$ if a, b, c, d are all integers or halves of odd integers.

Definition

The Norm of a quaternion $Q = a + bi + cj + dk$ is given by $Norm[Q] = a^2 + b^2 + c^2 + d^2$.

The Goal

Goal

Construct Greedy and maximally sized sets of quaternions of the Hurwitz Order free of three-term geometric progressions. For definiteness, we exclude progressions of the form

$$Q,\ \ QR,\ \ QR^2$$

where $Q, R \in \mathcal{H}$ and $Norm[R] \neq 1$.

Units and Factorization

Fact

The Hurwitz Order contains 24 units, namely

$$\pm 1, \pm i, \pm j, \pm k \text{ and } \pm \frac{1}{2} \pm \frac{1}{2} i \pm \frac{1}{2} j \pm \frac{1}{2} k.$$

Units and Factorization

Fact

The Hurwitz Order contains 24 units, namely

$$\pm 1, \pm i, \pm j, \pm k \text{ and } \pm \frac{1}{2} \pm \frac{1}{2} i \pm \frac{1}{2} j \pm \frac{1}{2} k.$$

Fact

Let Q be a quaternion of norm q. For any factorization of q into a product $p_0 p_1 \cdots p_k$ of integer primes, there is a factorization

$$Q = P_0 P_1 \cdots P_k$$

where P_i is a Hurwitz prime of norm p_i.

The Greedy Set

Recall Rankin's greedy set, G_3^*:
$1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21 \ldots$

The Greedy Set

Recall Rankin's greedy set, G_3^*:
$1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21\ldots$

Norms of elements in our greedy set:
$1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21\ldots$

The Greedy Set

Recall Rankin's greedy set, G_3^*:
$1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21 \ldots 48, 51 \ldots$

Norms of elements in our greedy set:
$1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21 \ldots$

The Greedy Set

Recall Rankin's greedy set, G_3^*:
$1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21 \ldots 48, 51 \ldots$

Norms of elements in our greedy set:
$1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21 \ldots 48, 49, 51 \ldots$

An Even Larger Set

$$S_N = \qquad \left(\frac{N}{4}, N\right]$$

An Even Larger Set

$$S_N = \left(\frac{N}{9}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$$

An Even Larger Set

$$S_N = \left(\frac{N}{24}, \frac{N}{12}\right] \cup \left(\frac{N}{9}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$$

An Even Larger Set

$$S_N = \left(\frac{N}{32}, \frac{N}{27}\right] \cup \left(\frac{N}{24}, \frac{N}{12}\right] \cup \left(\frac{N}{9}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$$

An Even Larger Set

$$S_N = \left(\frac{N}{40}, \frac{N}{36}\right] \cup \left(\frac{N}{32}, \frac{N}{27}\right] \cup \left(\frac{N}{24}, \frac{N}{12}\right] \cup \left(\frac{N}{9}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$$

An Even Larger Set

$$S_N = \left(\frac{N}{48}, \frac{N}{45}\right] \cup \left(\frac{N}{40}, \frac{N}{36}\right] \cup \left(\frac{N}{32}, \frac{N}{27}\right] \cup \left(\frac{N}{24}, \frac{N}{12}\right] \cup \left(\frac{N}{9}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$$

Acknowledgments

Co-collaborators:

- Eva Fourakis, Eli Goldstein, Steven J Miller, Nathan McNew, and Gwyn Moreland

Funding Sources

- NSF Grant DMS1347804
- Williams College
- Clare Boothe Luce Scholars Program

References

- N. McNew, *On sets of integers which contain no three terms in geometric progression*, Math. Comp., DOI: http://dx.doi.org/10.1090/mcom/2979 (2015).

- A. Best, K. Huan, N. McNew, S. J. Miller, J. Powell, K. Tor, M. Weinstein *Geometric-Progression-Free Sets Over Quadratic Number Fields*, arxiv.org/abs/1412.0999, (2014).

- R. A. Rankin, *Sets of integers containing not more than a given number of terms in arithmetical progression*, Proc. Roy. Soc. Edinburgh Sect. A **65** (1960/61), 332–344 (1960/61).