Optimizing test functions to bound the lowest zeros of cuspidal new forms

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Automorphic Forms Workshop
Denton, Texas, May 3, 2025

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) = \Lambda_\infty(s,f)L(s,f) = \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

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Modular Forms

Definition (Modular form of trivial nebentypus)

We write $f \in M_k(q)$ and say f is a *modular form* of level q, even weight k, and trivial nebentypus if $f : \mathbb{H} \to \mathbb{C}$ is holomorphic and

1. for each $\tau \in \Gamma_0(q) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{q} \right\}$ we have

$$f(\tau z) \ := \ f\left(\frac{az+b}{cz+d}\right) \ = \ (cz+d)^k f(z).$$

2. for $\tau \in \mathrm{SL}_2(\mathbb{Z})$, as $\mathfrak{Im}(z) \to +\infty$ we have $(cz+d)^{-k} f(\tau z) \ll 1$.

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With $\tau = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$, $f(z) = f(z+1)$ so f is 1-periodic and thus has a Fourier expansion at ∞:

$$f(z) = \sum_{n=0}^{\infty} a_f(n) q^n, \quad q = e^{2\pi i z}.$$

Modular Forms

Definition (Cuspform)

If $f \in M_k(q)$ vanishes at all cusps of $\Gamma_0(q)$ we say f is a *cuspform* and denote by $\mathcal{S}_k(q) \subset M_k(q)$ the space of holomorphic cuspforms.

Definition (Newform)

If f is an eigenform of *all* the Hecke operators and the Atkin-Lehner involutions $|_k W(q)$ and $|_k W(Q_p)$ for all the primes $p \mid q$, then we say that f is a *newform* and if, in addition, f is normalized so that $\psi_f(1) = 1$ we say that f is *primitive*.

Modular Forms

Definition (Cuspidal Newforms)

Let $H_k^\star(N)$ be the set of holomorphic cusp forms of weight k that are newforms of level N. For every $f \in H_k^\star(N)$, we have a Fourier expansion

$$f(z) \;=\; \sum_{n=1}^{\infty} a_f(n)e(nz)$$

with $e(z) = e^{2\pi iz}$. We set $\lambda_f(n) = a_f(n)n^{-(k-1)/2}$. The L-function associated to f is

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \lambda_f(n)n^{-s}.$$

Measures of Spacings: n-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

Definition

$$\mathscr{D}_n(f;\Phi) \ := \ \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_{i=1}^{n} \Phi_i\left(\frac{\gamma_f^{(j_i)}}{2\pi}\log c_f\right).$$

- Individual zeros contribute in limit.
- Most of contribution is from low zeros.
- Average over similar curves (family).

Normalization of Zeros

Local (hard, use c_f) vs Global (easier, use $\log c = |\mathcal{F}_N|^{-1} \sum_{f \in \mathcal{F}_N} \log c_f$).
Hope: ϕ a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i \left(\frac{\log c_f}{2\pi} \gamma_f^{(j_i)} \right)$$

$$\to \int \cdots \int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x) dx.$$

Katz-Sarnak Conjecture

As $c_f \to \infty$ the behavior of zeros near $1/2$ agrees with $N \to \infty$ limit of eigenvalues of a classical compact group.

1-Level Densities

The 1-level densities are conjecturally connected to the classical compact groups eigenvalues.

$$\begin{aligned} W_{1,O}(x) &= 1 + \frac{1}{2}\delta_0(x) \\ W_{1,\mathrm{SO(even)}}(x) &= 1 + \frac{\sin(2\pi x)}{2\pi x} \\ W_{1,\mathrm{SO(odd)}}(x) &= 1 - \frac{\sin(2\pi x)}{2\pi x} + \delta_0(x) \\ W_{1,Sp}(x) &= 1 - \frac{\sin(2\pi x)}{2\pi x} \end{aligned}$$

where $\delta_0(u)$ is the Dirac delta functional.

n-centered moments

Let $n \geq 2$ and $\mathrm{supp}(\phi) \subset (-\frac{\sigma}{n}, \frac{\sigma}{n})$. Define

$$\sigma_\phi^2 \ := \ 2\int_{-\infty}^{\infty} |y|\widehat{\phi}(y)^2 dy$$

and

$$R(m,i;\phi) \ := \ 2^{m-1}(-1)^{m+1}\sum_{l=0}^{i-1}(-1)^l\binom{m}{l} \left(-\frac{1}{2}\phi^m(0) + \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\widehat{\phi}(x_2)\cdots\widehat{\phi}(x_{l+1})\int_{-\infty}^{\infty}\phi^{m-l}(x_1)\frac{\sin(2\pi x_1(1+|x_2|+\cdots+|x_{l+1}|))}{2\pi x_1}dx_1\cdots dx_{l+1}\right)$$

and

$$S(n,a,\phi) \ := \ \sum_{l=0}^{\lfloor\frac{a-1}{2}\rfloor}\frac{n!}{(n-2l)!l!}R(n-2l,a-2l,\phi)\left(\frac{\sigma_\phi^2}{2}\right)^l.$$

By $\langle Q(f)\rangle_{N;\pm}$ we mean the average of $Q(f)$ over all f in the family of even (odd) cuspidal newforms of level N for the positive (negative) sign.

n-centered moments

We may consider an equivalent definition of n-level densities

Theorem (Cohen, et al. '22)

Assume GRH for Dirichlet L-functions and for cuspidal newforms and their symmetric squares. Then for $\sigma_n = 2$,

$$\lim_{\substack{N\to\infty \\ N \text{prime}}} \langle (D(f;\phi) - \langle D(f;\phi)\rangle_{N;\pm})^n \rangle_{N;\pm} \;=\; 1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n,a;\phi), \qquad (1)$$

where

$$1_{n \text{ even}} \;:=\; \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Remarks on Computation and Support σ

- Improving σ allows test functions to more accurate compare average density to RMT
- Calculating statistics based on known data, such as non-vanishing
- Improving statistics with optimal Test Functions

Results

Previous Results

Question

Assuming the GRH, how far up must we go on the critical line before we are assured that we will see the first zero?

Previous work mostly on first (lowest) zero of an L-function. Assume GRH, zeros of the form $\frac{1}{2} + i\gamma$.

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Assuming the GRH, how far up must we go on the critical line before we are assured that we will see the first zero?

Previous work mostly on first (lowest) zero of an L-function. Assume GRH, zeros of the form $\frac{1}{2} + i\gamma$.

- S. D. Miller: L-functions of real archimedian type has $\gamma < 14.13$.
- J. Bober, J. B. Conrey, D. W. Farmer, A. Fujii, S. Koutsoliotas, S. Lemurell, M. Rubinstein, H. Yoshida: General L-function has $\gamma < 22.661$.

New Results:

Theorem: Upper Bound Lowest First Zero in Even Cuspidal Families

For an odd n, whenever ω satisfies this following inequality

$$-\left(\widehat{\phi_\omega}(0)+\frac{1}{2}\int_{-\sigma/n}^{\sigma/n}\widehat{\phi_\omega}(y)dy\right)^n < 1_{n\text{ even}}(n-1)!!\sigma_{\phi_\omega}^n + S(n,a;\phi_\omega),$$

at least one form with at least one normalized zero in $(-\omega,\omega)$. Consequently, if

$$\omega > \left(-\frac{\sigma\int_0^1 h(u)^2\,du+\frac{\sigma^2}{4}\int_0^{2/\sigma}\int_{v-1}^1 h(u)h(v-u)\,du\,dv}{\frac{1}{\sigma}\int_0^1 h(u)h''(u)\,du+\frac{1}{4}\int_0^{2/\sigma}\int_{v-1}^1 h(u)h''(v-u)\,du\,dv}\right)^{-\frac{1}{2}}\pi^{-1}, \tag{2}$$

then at least one form with at least one normalized zero in $(-\omega,\omega)$.

New Results

Theorem: Normalized Zeros Near the Central Point

$P_{r,\rho}(\mathcal{F})$: percent of forms with at least r normalized zeros in $(-\rho, \rho)$.

For even n and $r \geq \mu(\phi, \mathcal{F})/\phi(\rho)$:

$$P_{r,\rho}(\mathcal{F}) \ \leq \ \frac{1_{n \text{ even}}(n-1)!!\sigma_\phi^n + S(n, a; \phi)}{(r\phi(\rho) - \mu(\phi, \mathcal{F}))^n},$$

where $\sigma_\phi = \sqrt{2 \int_{\mathbb{R}} |y|\widehat{\phi}(y)^2 dy}$.

Explicit Bounds

Naive Test Function

The naive test functions are the Fourier pair

$$\phi_{\text{naive}}(x) = \left(\frac{\sin(\pi\sigma_n x)}{(\pi\sigma_n x)}\right)^2 \quad , \quad \widehat{\phi}_{\text{naive}}(y) = \frac{1}{\sigma_n}\left(y - \frac{|y|}{\sigma_n}\right)$$

for $|y| < \sigma_n$ where σ_n is the support.

Explicit Bounds

Number of zeros	2-level	4-level	6-level
6	N/A	10.849910	48.154279
16	N/A	0.004235	$2.83230 \cdot 10^{-4}$
26	N/A	$3.541901 \cdot 10^{-4}$	$6.716802 \cdot 10^{-6}$
28	420.045063	$2.486819 \cdot 10^{-4}$	$3.943864 \cdot 10^{-6}$
30	20.991406	$1.796948 \cdot 10^{-4}$	$2.418466 \cdot 10^{-6}$
32	6.651738	$1.330555 \cdot 10^{-4}$	$1.538761 \cdot 10^{-6}$
34	3.220871	$1.006126 \cdot 10^{-4}$	$1.010576 \cdot 10^{-6}$

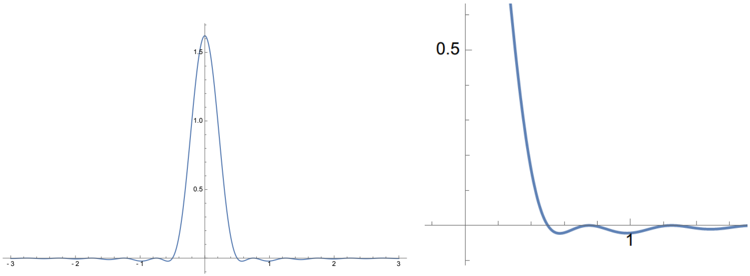
Table: Upper bound on percentage of forms with at least r normalized zeros within 0.8 average spacing from central point, using naive test function with support $2/n$. "N/A" means restriction in our theorem not met.

Constructions and Proofs

Construction of Test Function

Create compactly supported $\widehat{\phi}(y)$.

- Choose $h(y)$ even, twice continuously differentiable, supported on $(-1, 1)$, monotonically decreasing.
- $f(y) := h\left(\frac{2y}{\sigma/n}\right)$.
- $g(y) := (f * f)(y), \quad \widehat{g}(x) = \widehat{f}(x)^2 \geq 0$.
- $\widehat{\phi}_\omega(y) := g(y) + (2\pi\omega)^{-2} g''(y)$ thus $\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2)$.



Plot of $\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2)$, for $h = \cos\left(\frac{\pi y}{2}\right)$, $\sigma = 2$, $\omega = .5$, and $n = 1$.

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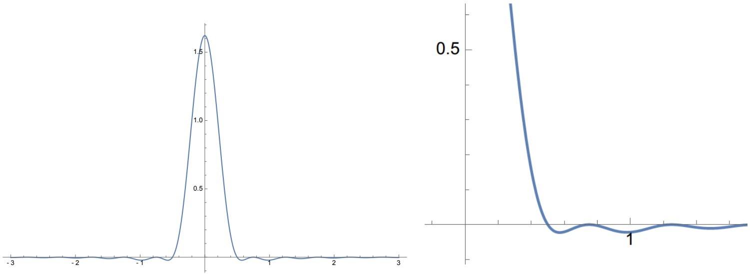

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Sketch of Proof: Key Expansion

Theorem: Upper Bound Lowest First Zero in Even Cuspidal Families

For odd n, whenever ω satisfies this following inequality

$$-\left(\widehat{\phi_\omega}(0) + \frac{1}{2}\int_{-\sigma/n}^{\sigma/n} \widehat{\phi_\omega}(y)dy\right)^n < 1_{n \text{ even}}(n-1)!!\sigma_{\phi_\omega}^n + S(n, a; \phi_\omega),$$

there exists at least one form with at least one normalized zero in $(-\omega, \omega)$.

Sketch of Proof: Key Expansion

Replace mean from finite N with the limit:

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} \frac{1}{|\mathscr{F}_N|} \sum_{f\in\mathscr{F}_N} \left(\sum_j \phi\left(\widetilde{\gamma}_{f,j}\right) - \mu(\phi, \mathscr{F}) \right)^n$$
$$= 1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n, a; \phi),$$

where the mean of the 1-level density of $\mathscr{F}_N$ is

$$\mu(\phi, \mathscr{F}) \ := \ \widehat{\phi}(0) + \frac{1}{2}\int_{-\infty}^{\infty} \widehat{\phi}(y)dy.$$

Key Observation

$$\lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \left(\sum_j \phi(\widetilde{\gamma}_{f,j}) - \mu(\phi, \mathcal{F}) \right)^n$$
$$= 1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n, a; \phi).$$

$$\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2).$$

- $\phi_\omega(x) \geq 0$ when $|x| \leq \omega$, and $\phi_\omega(x) \leq 0$ when $|x| > \omega$.
- Contribution of zeroes for $|x| \geq \omega$ is non-positive.
- As n odd, doesn't decrease if drop these non-positive contributions: why we restrict to odd n.

Sketch of Proof: Proof by Contradiction

Dropping negative contributions:

$$\lim_{\substack{N\to\infty \\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \left(\sum_{|\widetilde{\gamma}_{f,j}|\leq\omega} \phi_\omega(\widetilde{\gamma}_{f,j}) - \mu(\phi_\omega, \mathcal{F}) \right)^n \geq S(n, a; \phi_\omega).$$

Sketch of Proof: Proof by Contradiction

Assume no forms have a zero on the interval $(-\omega, \omega)$:

$$\lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} (-\mu(\phi_\omega, \mathcal{F}))^n \ \geq\ S(n, a; \phi_\omega),$$

$$(-\mu(\phi_\omega, \mathcal{F}))^n \lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} 1 \ \geq\ S(n, a; \phi_\omega).$$

As $\lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} 1 = 1$, get

$$(-\mu(\phi_\omega, \mathcal{F}))^n \ \geq\ S(n, a; \phi_\omega).$$

Sketch of Proof: Continued

Because of the compact support of $\widehat{\phi}_\omega$,

$$-\left(\widehat{\phi}_\omega(0) + \frac{1}{2}\int_{-\sigma/n}^{\sigma/n} \widehat{\phi}_\omega(y)dy\right)^n \geq S(n, a; \phi_\omega).$$

Thus, if ω satisfies the following inequality

$$-\left(\widehat{\phi}_\omega(0) + \frac{1}{2}\int_{-\sigma/n}^{\sigma/n} \widehat{\phi}_\omega(y)dy\right)^n < S(n, a; \phi_\omega),$$

we get a contradiction, so at least one form has a normalized zero in $(-\omega, \omega)$.

Explicit Bound from 1-Level Density

First Zero from 1-Level

The first zero of the family of cuspidal newforms exists on the interval $(-\omega_{\min}, \omega_{\min})$, where

$$\omega_{\min} > \left(- \frac{\sigma \int_0^1 h(u)^2 \, du + \frac{\sigma^2}{4} \int_0^{2/\sigma} \int_{v-1}^{1} h(u) h(v-u) \, du \, dv}{\frac{1}{\sigma} \int_0^1 h(u) h''(u) \, du + \frac{1}{4} \int_0^{2/\sigma} \int_{v-1}^{1} h(u) h''(v-u) \, du \, dv} \right)^{-\frac{1}{2}} \pi^{-1}. \quad (3)$$

Number theory known only for $\sigma < 2$ (under GRH).

For $h(y) = \cos(\pi y/2)$, we obtain $\omega_{\min}(2, h) > 0.21864$.

Main Theorem 2

Theorem: Normalized Zeros Near the Central Point

$P_{r,\rho}(\mathscr{F})$: percent of forms with at least r normalized zeros in $(-\rho, \rho)$.
For even n and $r \geq \mu(\phi, \mathscr{F})/\phi(\rho)$:

$$P_{r,\rho}(\mathscr{F}) \ \leq \ \frac{1_{n \text{ even}}(n-1)!!\sigma_\phi^n + S(n, a; \phi)}{(r\phi(\rho) - \mu(\phi, \mathscr{F}))^n}.$$

Sketch of Proof

Even n, dropping all with less than r zeros in $(-\rho, \rho)$ drops a non-negative sum:

$$\lim_{\substack{N\to\infty \\ N\text{prime}}} \frac{1}{|\mathscr{F}_N|} \sum_{f\in\mathscr{F}_{N,r}^{(\rho)}} \left(\sum_{|\widetilde{\gamma}_{f,j}|<\rho} \phi(\widetilde{\gamma}_{f,j}) + T_f(\phi) - \mu(\phi,\mathscr{F}) \right)^n \ \le \ 1_{n \text{ even}}(n-1)!!\sigma_\phi^n + S(n,a;\phi)$$

Replace the summation of $\phi(\widetilde{\gamma}_{f,j})$ with $r\phi(\rho)$; can drop $T_f(\phi)$ and not increase LHS if $r \ge \mu(\phi,\mathscr{F})/\phi(\rho)$:

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} \frac{1}{|\mathscr{F}_N|} \sum_{f\in\mathscr{F}_{N,r}^{(\rho)}} (r\phi(\rho) - \mu(\phi,\mathscr{F}))^n \ \le \ \dots.$$

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Explicit Bounds

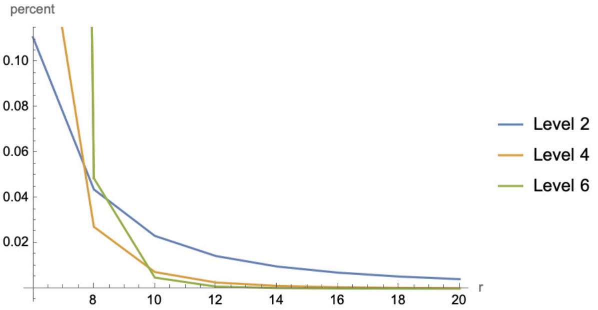

Figure: Percentage vs. number of zeros (for a fixed $\rho = .4$).

Higher levels starts above lower when r small, decrease faster and eventually gives better results as r grows.

Expanding Space for Test Functions

Main Idea

The construction of the test function requires $\widehat{g}(x)$ to decay at the rate of $\Theta(|x|^{-4})$ so it may decay faster than the term $(1-(x/\omega)^2)$.

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Main Idea

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We can multiply $\phi(x)$ by a polynomial term of an even degree such that $\widehat{g}(x)$ decays at a rate $|x|^{-A}$, where $A > 4$.

Thus, we may consider a larger space of polynomial, that we may optimize with a program with respect to.

Conditions on the Polynomial

As mentioned previously, ϕ_ω must satisfy the condition, such that $\phi_\omega(x) \geq 0$ when $|x| \leq \omega$ and $\phi_\omega \leq 0$ when $|x| > \omega$ and must be even and decay, such that $\phi_\omega \to 0$ as $x \to \infty$.

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Therefore the polynomial term must be positive and even, so we can write

$$\phi(x) = \widehat{g}(x)(1 - (x/\omega)^2)(1 + c_1 x^2 + c_2 x^4 + \ldots + c_w x^{2w}),$$

where w is the degree of differentiability of $h(x)$ at $x = 1$.

Since $\widehat{g}_w(x) = \widehat{g}(x)(1 + c_1 x^2 + c_2 x^4 + ... + c_w x^{2w})$,

$$\widehat{g}_w(x) = \widehat{g}(x) + c_1 \widehat{g}(x) x^2 + c_2 \widehat{g}(x) x^4 + \cdots + c_w \widehat{g}(x) x^{2w}).$$

We then use the properties of the Fourier transform to deduce that

$$\begin{aligned} g_w(x) &= g(x) - c_1 (2\pi)^{-2} g''(x) + \cdots + c_w (2\pi i)^{-2w} \frac{d^{2w}}{dx^{2w}} g(x) \\ &= g(x) + \sum_{k=1}^{w} c_k (-4\pi^2)^{-k} \frac{d^{2k}}{dx^{2k}} g(x). \end{aligned}$$

New Result

From the same methods used to prove the original bound on the first zero for even families, we obtain,

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From the same methods used to prove the original bound on the first zero for even families, we obtain,

$$\omega_{\min} > \frac{1}{2\pi}\left(-\frac{g_w''(0)+\int_0^1 g_w''(x)\,dx}{\int_0^1 g_w(x)\,dx+g_w(0)}\right)^{1/2}.$$

Constraints on Coefficients

We can consider the constraints on the coefficients c_k of the polynomial. Consider

$$p_a(x) = \prod_{i=1}^{a} (\mu_i x^2 - 1)^2,$$

a positive even polynomial of degree $4a$ with all real roots.

The c_k terms depend on roots λ_i parameters so we write,

$$c_k \;=\; (-1)^{2a-k} \sum_{1 \le r_1 < r_2 < \cdots < r_i \le 2a} \lambda_{r_1} \lambda_{r_2} \cdots \lambda_{r_i}.$$

Because all the zeros are real, the coefficients c_k of p_a are minimal constants.

Since we aim to minimize $\omega_{\min}$ with respect to the c_k we use a program to minimize the $\{\lambda_i\}$ given w, h. Take

$$h(x) \;=\; (1-x^2)^{2w+1}\left(\prod_{j=1}^{s}(1-\alpha_j x^2)+\beta\right),$$

where s denotes the number of zeros this polynomial may have and $0 \leq \alpha_j \leq 1$ and $\beta \geq 0$.

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Thus, a minimization program may be able to take in the constants of σ, s, and w, while optimizing constraints for α_j and λ_i to minimize ω with respect to these parameters.

When letting the differentiability of h, $w = 1$, the support of the test function, $\sigma = 2$, and the degree of the polynomial for h,$s = 4$, a Mathematica program suited for minimization estimates $\omega_{\text{min}} = 0.218503$.

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There is a convergence of c_k independent of the of the original $h(x)$, so the zeros of an optimal g_ω may be approximated by a program

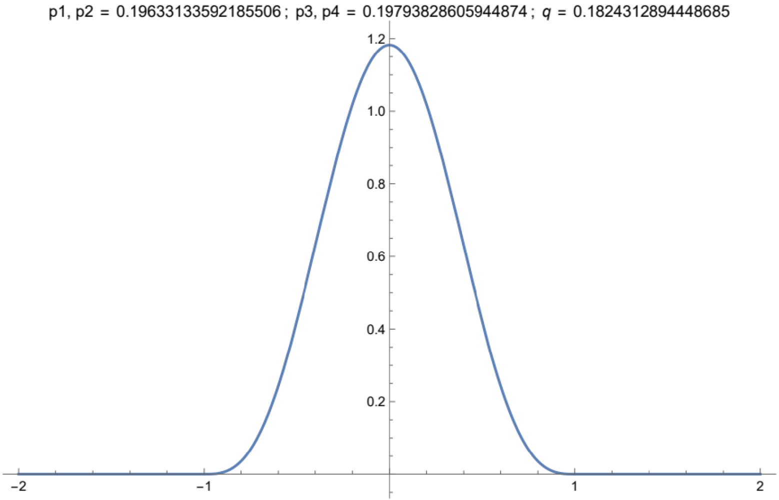

Figure: Result of a program optimizing h for $w, \sigma, s = 1, 2, 4$ respectively.

Future Works

Improving Bounds

- Generalizing Test Function Construction and Program
- Increase support of test function.
- Recent studies increased the support to 4 (Baluyot, Chandee, and Li) for a certain group of L-functions....

Acknowledgments and References

Acknowledgments

This work was supported by NSF Grant DMS2241623, Williams College, The Finnerty Fund, and the Winston Churchill Foundation. We thank the organizers of the 37th Automorphic Forms Workshop for the opportunity to speak today.

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