Learning Spheres and Chains in $\mathbb{F}_q^d$

Ryan Jeong, Wyatt Milgrim

Joint work with Ruben Ascoli, Livia Betti, Justin Cheigh, Xuyan Liu, Brian McDonald, Francisco Romero, and Santiago Velazquez; Advisors: Alex Iosevich and Steven J. Miller

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- For the remainder of the talk we let t be a fixed, nonzero element of the finite field $\mathbb{F}_q$ (i.e. this is not a variable).

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The *Vapnik–Chervonenkis dimension* (VCD) of $\mathcal{H}$ is the cardinality of the largest subset $A \subset E$ that is shattered by $\mathcal{H}$.

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Explicitly, $VCD(\mathcal{H}) = n$ if there exists $A \subset E$, $|A| = n$ such that A is shattered by $\mathcal{H}$, but no such subset of size $n + 1$.

VC-Dimension: Example

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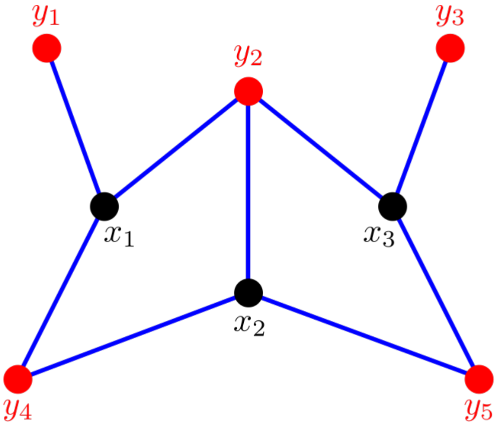

Figure: $E = \{x_1, x_2, x_3\}$ and classifiers in $\mathcal{H}$ correspond to spheres of some fixed radius centered at points y_i. We can shatter $A = \{x_1, x_2\} \subset E$: centers $\{y_1, y_3, y_4, y_5\}$ yield all four possible behaviors. Here, $\text{VCD}(\mathcal{H}) = 2$, since there is no y_i adjacent to strictly x_1 and x_3.

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Motivated by these seminal results, we can study *learning natural families of classifiers over finite fields.*

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- However, the theorem does not generalize easily to higher dimensions (i.e. the task of learning spheres $\mathbb{F}_q^d$ with $d \geq 3$).

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Note that we have the bound VCD $\left(\mathcal{H}_t^d(E)\right) \leq d$, since $d+1$ points determine a sphere in $\mathbb{F}_q^d$.

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The $(d+2)$-tuple $P = (y, z, x^1, \ldots, x^d) \in (\mathbb{F}_q^d)^{d+2}$ is an *d-prism* if for all $i \leq n$, $||x^i - y|| = ||x^i - z|| = t$.

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We say a prism is *non-degenerate* if all of its points are distinct.

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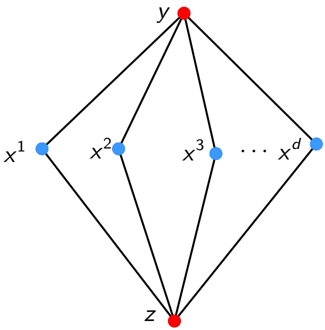

Figure: A nondegenerate d-prism $P = (y, z, x^1, \ldots, x^d)$ with tail $\mathcal{T}(P) = \{y, z\}$ and center $\mathcal{C}(P) = \{x^1, \ldots, x^d\}$. A d-prism can be thought of as possessing the subgraph $K_{2,d}$, with vertices in the edge $\{v, w\}$ edge satisfying $||v - w|| = t$.

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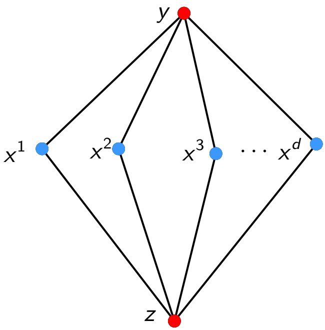

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Henceforth, the term prism refers to a nondegenerate d-prism.

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- *To shatter the center* $\mathcal{C}(P)$ *of a prism* P*, for every subset* $A \subset \mathcal{C}(P)$*, we must find a point distance* t *away only from the points in* A*.*

Proof Strategy

Definition (SMALL 2022++)

Fix a prism $P = (\mathcal{T}, \mathcal{C})$ with center $\mathcal{C}$. A subset $A \subset \mathcal{C}$ with $|A| \geq 2$ is P-bad if every $x \in \text{Pole}(A)$ is also in $\text{Pole}(y)$ for some $y \in \mathcal{C} \setminus A$.

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- Count the number of prisms, denoted $N_d(E)$.
- Count the number of prisms that can contain a bad k-set by counting the number of prisms a given k-set can be bad in.
- Show that, if $|E|$ is large, there must be some prism for which no subset of its center is bad.

Prisms, Pairs, and 2-Paths

Definition

A 2-path in E is a set $(x_1, x_2, x_3) \in E^3$ such that $||x_1 - x_2|| = ||x_2 - x_3|| = t$.

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Up to ordering a prism corresponds to a choice of pair (y, z) and a choice of d 2-paths between them—(y, z) are the tails and the set of midpoints of these paths is the center (there is some concern regarding degenerate prisms but this is easy to deal with).

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Let $E \subset \mathbb{F}_q^d$ with $|E| > \frac{4}{\ln(2)} q^{\frac{d+1}{2}}$ and let $\Gamma_2(E)$ be the number of 2-paths in E. Then $\Gamma_2(E) = \frac{|E|^3}{q^2} + \mathcal{D}_2(E)$ where $\mathcal{D}_2(E) \leq C \frac{|E|^2}{q}$.

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- From here it is straightfoward to apply Hölder's Inequality (proof omitted) to obtain $N_d(E) \geq C\frac{|E|^{d+2}}{q^{2d}}$

Finding a Good Prism

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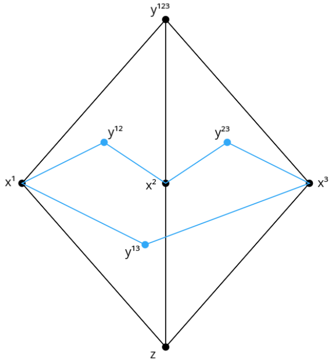

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- Finally, observe that there are only $1/2|E|^2$ pairs total, and so at most $C|E|^2 q$ prisms admit a P-bad set. But we just showed there are at least $|E|^5 q^{-6}$ prisms total so we obtain the following.

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- However, for the proof to work we need that the maximum number of poles a k-bad set can have is order q^{d-k-1}. This is always the case if we restrict our attention to prisms with affinely independent centers. We were able to show that enough such prisms exist if $|E| > q^{d-\frac{1}{d-1}}$.

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- Our counting argument gets more complicated in higher dimensions since there are no longer only 2 poles for each bad set, and thus there can be more than one choice of tails for prisms a set can be bad in.
- However, if a bad set has additional poles, then we have additional constraints on prisms it can be bad in since each such pole must also be a pole of some other center point in each prism. This counterbalances the additional choices of tails to an extent.
- However, for the proof to work we need that the maximum number of poles a k-bad set can have is order q^{d-k-1}. This is always the case if we restrict our attention to prisms with affinely independent centers. We were able to show that enough such prisms exist if $|E| > q^{d-\frac{1}{d-1}}$.
- This is the strongest bound on $|E|$ needed in our proof so we obtain our main result.

Future Work

One direction would be attempting to extend past work on single-sphere classifiers. This direction is very similar to what we studied, but is much harder since the greatest VC dimension possible is $d+1$ instead of d.

Conjecture (Fitzpatrick, Iosevich, McDonald, and Wyman, 2021)

For large subsets $E \subset \mathbb{F}_q^d$, then VCD $\left(\mathcal{H}_t^d(E)\right) = d+1$, where $\mathcal{H}_t^d(E)$ denote sphere classifiers over the subset E.

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It might also be interesting to pursue the chain direction further.

Problem (SMALL 2022++)

Can we get an analogous result for longer chains? How would such classifiers even be defined?

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