Introduction to Cryptography: Alphabet Codes

Introduction to Cryptography: Alphabet Codes: Steven J. Miller

http://www.williams.edu/Mathematics/sjmiller/public_html

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Caesar and Affine Ciphers

Encryption Functions

Function from alphabet to alphabet such that:

- Different letters go to different letters.
- All letters hit.

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Setup: x input letter, $f(x)$ is the encrypted letter.

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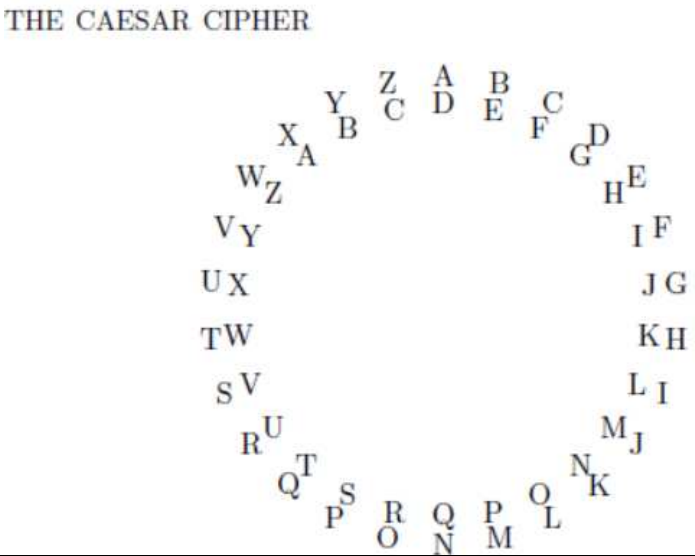
- $c = 3$ is the standard Caesar cipher: how many choices?
- $A \to D, B \to E, \ldots, W \to Z, X \to A, Y \to B, Z \to C$ (clock arithmetic).
- How do we decrypt?

Caesar Cipher: Illustration

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C

Caesar Cipher: Illustration



Caesar Cipher: Example

	M	E	E	T	A	T	T	E	N
	12	4	4	19	0	19	19	4	13
add 3:	15	7	7	22	3	22	22	7	16
	P	H	H	W	D	W	W	H	Q

	P	H	H	W	D	W	W	H	Q
	15	7	7	22	3	22	22	7	16
subtract 3:	12	4	4	19	0	19	19	4	13
	M	E	E	T	A	T	T	E	N

Affine Cipher: $f(x) = ax + b$

Try more complicated function: $f(x) = ax + b$.

What b are possible?

Often best to start simple: take 6 letter alphabet: $\{A, B, C, D, E, F\}$.

To do calculations let $A = 1$, $B = 2$, $\ldots$, $F = 6$.

Do all b work? Do all a work?

Affine Cipher: II: $f(x) = ax + b$

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$$\begin{array}{ccccccccccc} A & +1 \Rightarrow & B & +1 \Rightarrow & C & +1 \Rightarrow & D & +1 \Rightarrow & E & +1 \Rightarrow & F \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ C & +1 \Rightarrow & D & +1 \Rightarrow & E & +1 \Rightarrow & F & +1 \Rightarrow & A & +1 \Rightarrow & B \end{array}$$

$$\begin{array}{ccccccccccc} A & +1 \Rightarrow & B & +1 \Rightarrow & C & +1 \Rightarrow & D & +1 \Rightarrow & E & +1 \Rightarrow & F \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ C & +2 \Rightarrow & E & +2 \Rightarrow & A & +2 \Rightarrow & C & +2 \Rightarrow & E & +2 \Rightarrow & A \end{array}$$

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- $a = 1$: works: get $\{1, 2, 3, 4, 5, 6\}$.
- $a = 2$: fails: get $\{2, 4, 6, 2, 4, 6\}$ (thus 4 and 6 will also fail).
- $a = 3$: fails: get $\{3, 6, 3, 6, 3, 6\}$ (and also get again 6 fails).
- $a = 5$: works: get $\{5, 4, 3, 2, 1, 6\}$.

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- $a = 5$: works: get $\{5, 4, 3, 2, 1, 6\}$.

1, 5 work and 2, 3, 4, 6 fail: What's the pattern?

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Alphabet is $A = 1, \ldots, F = 6$.

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What do you think works for 26 letter alphabet?

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What do you think works for 26 letter alphabet?

Answer: 1, 3, 5, 7, 9, 11, 15, 17, 19, 21, 23, 25: 12 choices.

Caesar cipher: 26 choices; Affine cipher: $12 \cdot 26 = 312$.

Can we go to higher degree polynomials?

Totient Function

Will use the Euler totient function $\phi(n)$ later.
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If $n = pq$ are distinct primes then $\phi(pq) = (p-1)(q-1)$:

- Lose $p, 2p, 3p, \ldots, qp$.
- Lose $q, 2q, 3q, \ldots, pq$.
- Double counted pq so add back: $pq - q - p + 1 = (p-1)(q-1)$.

Vigenére, and Permutation Ciphers

Vigenére

Issue is always send a letter to same new letter....

Take a keyphrase, write under and use for shifts:

D A D C A N A D D A B E D

A B C A B C A B C A B C A

E C G D C Q B F G B D H E

So D shifted to E, F, G; A shifted to B, C; ....
How secure is this?

Vigenére

How secure is the Vigenére cipher?

- https://www.telegraph.co.uk/news/worldnews/northamerica/usa/8225871/CIA-decodes-Civil-War-message-in-a-bottle-after-147-years.html
- https://owlcation.com/humanities/1863-Siege-of-Vicksburg-Secret-Message-Decoded
- http://archive.boston.com/news/nation/articles/2010/12/26/civil_war_note_finally_deciphered/
- http://cryptiana.web.fc2.com/code/civilwar4.htm
- https://en.wikipedia.org/wiki/Kasiski_examination

Permutation Ciphers

26 choices for A, 25 for B, ...

$26! \approx$

Permutation Ciphers

26 choices for A, 25 for B, ...

$26! \approx 4 \cdot 10^{26}$.

A lot more than affine case!

How secure are these?

Frequency Attacks

1	e	12.58%	14	m	2.56%
2	t	9.09%	15	f	2.35%
3	a	8.00%	16	w	2.22%
4	o	7.59%	17	g	1.98%
5	i	6.92%	18	y	1.90%
6	n	6.90%	19	p	1.80%
7	s	6.34%	20	b	1.54%
8	h	6.24%	21	v	0.98%
9	r	5.96%	22	k	0.74%
10	d	4.32%	23	x	0.18%
11	l	4.06%	24	j	0.15%
12	u	2.84%	25	q	0.12%
13	c	2.58%	26	z	0.08%

TABLE 1. Frequencies of letters in English text, from 9,481 English works from Project Gutenberg; see http://www.cryptograms.org/letter-frequencies.php.

Frequency Attacks

most common bigrams	
1	th
2	he
3	in
4	en
5	nt
6	re
7	er
8	an
9	ti
10	es
11	on
12	at
13	se
14	nd
15	or
16	ar
17	al

common trigrams	
1	the
2	and
3	tha
4	ent
5	ing
6	ion
7	tio
8	for
9	nde
10	has
11	nce
12	edt
13	tis
14	oft
15	sth

Why Primes?

Two Systems

p, q are 200 digit primes, $N = pq$ public, password p **or** q.

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Say every atom in the universe (about 10^{80} such) is a universe, and each atom there is a supercomputer checking 10^{15} items a second. About $3.2 \cdot 10^7$ seconds in a year, check about $3.2 \cdot 10^{182}$ per year.

Universe about 15 billion years old, so in the life of the universe would check about $5 \cdot 10^{192}$. Less than the number of primes to check!

RSA Description
(Rivest, Shamir, and Adleman)

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Bob does the following (could have b subscripts):

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- Decrypt: $X^d \bmod N$ or 195632041.

Imagine receive $\widetilde{X} = 121209483$.
Message 195632041
Decrypts 121141028, only two digits are the same!

Implementation Questions

A lot of implementation issues.

- How do we find large primes? How large is large?
- How do we find e and d so that $ed = 1 \bmod (p-1)(q-1)$?
- How do we compute $M^e \bmod N$ efficiently?
- Can Eve determine d from e and N?