Learning Spheres and Chains in $\mathbb{F}_q^d$

Wyatt Milgrim

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Preliminaries

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- For the remainder of the talk we let t be a fixed, nonzero element of the finite field $\mathbb{F}_q$ (i.e. this is not a variable).

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Definition (Vapnik and Chervonenkis, 1968)

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Explicitly, $VCD(\mathcal{H}) = n$ if there exists $A \subset E$, $|A| = n$ such that A is shattered by $\mathcal{H}$, but there is no such subset of size $n+1$.

VC-Dimension: Example

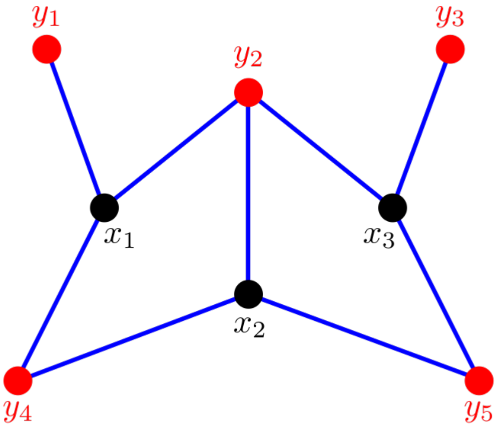

Figure: $E = \{x_1, x_2, x_3\}$ and classifiers in $\mathcal{H}$ correspond to spheres of some fixed radius centered at points y_i. We can shatter $A = \{x_1, x_2\} \subset E$: centers $\{y_1, y_3, y_4, y_5\}$ yield all four possible behaviors. Here, $\text{VCD}(\mathcal{H}) = 2$, since there is no y_i adjacent to strictly x_1 and x_3.

Prior Work: Learning Spheres in $\mathbb{F}_q^2$

Spheres in $\mathbb{F}_q^2$: Fix $t \neq 0$. For $E \subset \mathbb{F}_q^2$, define the class of functions $\mathcal{H}_t^2(E) = \{h_y : y \in E\}$, where $h_y : E \to \{0,1\}$ is the indicator function for the sphere of radius t centered at y:

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- Three points determine a circle in $\mathbb{F}_q^2$, so VCD $\left(\mathcal{H}_t^2(E)\right) \leq 3$. More generally, VCD $\left(\mathcal{H}_t^d(E)\right) \leq d+1$.

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- Three points determine a circle in $\mathbb{F}_q^2$, so $\text{VCD}\left(\mathcal{H}_t^2(E)\right) \leq 3$. More generally, $\text{VCD}\left(\mathcal{H}_t^d(E)\right) \leq d + 1$.
- However, the theorem does not generalize easily to higher dimensions (i.e. the task of learning spheres $\mathbb{F}_q^d$ with $d \geq 3$).

New Classifiers: Learning 2-Chains in $\mathbb{F}_q^d$

2-Chains in $\mathbb{F}_q^d$: Fix $t \neq 0$, $d \geq 3$, and $E \subset \mathbb{F}_q^d$. Define the collection of functions $\mathcal{H}_t^d(E) = \{h_{y,z} : y, z \in E, y \neq z\}$, where $h_{y,z} : E \to \{0, 1\}$ is the indicator function for the intersection of spheres of radius t centered at y and z:

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Theorem (SMALL 2022++)

If $|E| \geq \begin{cases} Cq^{\frac{7}{3}} & d = 3 \\ Cq^{d-\frac{1}{d-1}} & d \geq 4 \end{cases}$

Where C depends on d but not q, then $VCD\left(\mathcal{H}_t^d(E)\right) = d$.

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Note that we have the bound $\text{VCD}\left(\mathcal{H}_t^d(E)\right) \leq d$, since $d+1$ points determine a sphere in $\mathbb{F}_q^d$.

Definitions

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Fix $E \subset \mathbb{F}_q^d$, and let $A \subset E$. Call $x \in E$ a *pole* of A if $||x - a|| = t$ whenever $a \in A$, and let $\text{Pole}(A) \subset E$ denote the set of poles of A.

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Definition

The $(d+2)$-tuple $P = (y, z, x^1, \ldots, x^d) \in (\mathbb{F}_q^d)^{d+2}$ is an *d-prism* if for all $i \leq n$, $||x^i - y|| = ||x^i - z|| = t$.

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We say a prism is *non-degenerate* if all of its points are distinct.

Prisms: Distance Graph

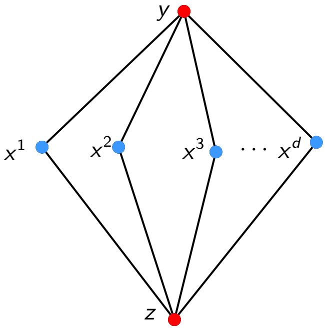

Figure: A nondegenerate d-prism $P = (y, z, x^1, \ldots, x^d)$ with tail $\mathcal{T}(P) = \{y, z\}$ and center $\mathcal{C}(P) = \{x^1, \ldots, x^d\}$. A d-prism can be thought of as being the graph $K_{2,d}$, with vertices as its points and edges $\{v, w\}$ when $||v - w|| = t$.

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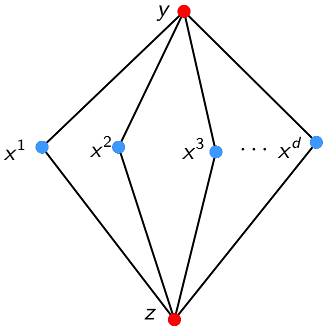

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Henceforth, the term prism refers to a nondegenerate d-prism.

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- *To shatter the center* $\mathcal{C}(P)$ *of a prism* P*, for every subset* $A \subset \mathcal{C}(P)$*, we must find a point distance* t *away only from the points in* A*.*

Proof Strategy

Definition (SMALL 2022++)

Fix a prism $P = (\mathcal{T}, \mathcal{C})$ with center $\mathcal{C}$. A subset $A \subset \mathcal{C}$ is P-bad if every $x \in \mathrm{Pole}(A)$ is also in $\mathrm{Pole}(y)$ for some $y \in \mathcal{C} \setminus A$.

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- Count the number of prisms, denoted $N_d(E)$.
- Count the number of prisms that can contain a bad k-set by counting the number of prisms a given k-set can be bad in.
- Show that, if $|E|$ is large, there must be some prism for which no subset of its center is bad.

Prisms, Pairs, and 2-Paths

Definition

A 2-path in E is a set $(x_1, x_2, x_3) \in E^3$ such that $||x_1 - x_2|| = ||x_2 - x_3|| = t$.

Observation

Up to ordering a prism corresponds to a choice of pair (y, z) and a choice of d 2-paths between them—(y, z) are the tails and the set of midpoints of these paths is the center (there is some concern regarding degenerate prisms but this is easy to deal with).

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Let $E \subset \mathbb{F}_q^d$ with $|E| > \frac{4}{\ln(2)} q^{\frac{d+1}{2}}$ and let $\Gamma_2(E)$ be the number of 2-paths in E. Then $\Gamma_2(E) = \frac{|E|^3}{q^2} + \mathcal{D}_2(E)$ where $\mathcal{D}_2(E) \leq C\frac{|E|^2}{q}$.

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$$N_d(E) = d! \sum_{(x,y)\in E\times E} \binom{k_{(x,y)}}{d}$$

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- From here it is straightfoward to apply Hölder's Inequality (proof omitted) to obtain $N_d(E) \geq C\frac{|E|^{d+2}}{q^{2d}}$

The Proof for $d = 3$

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In the $d = 3$ case we say that a pair $\{a, b\}$ is P-bad for a prism $P = \{(y, z, a, b, x)\}$ if $\text{Pole}(\{a, b\}) \subset \text{Pole}(\{a, b, x\})$. We say a pair is **bad** if it is P-bad for some prism P.

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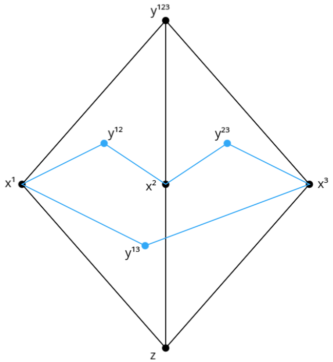

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- Finally, observe that there are only $1/2|E|^2$ pairs total, and so at most $C|E|^2 q$ prisms admit a P-bad pair. But we just showed there are at least $|E|^5 q^{-6}$ prisms total so we obtain the following.

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Let $d = 3$ and suppose $|E| \geq Cq^{7/3}$. Then there exists a prism that admits no P-bad pairs, and thus the VC-dimension of $\mathcal{H}_t^3(E)$ equals 3.

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- This is the strongest bound on $|E|$ needed in our proof so we obtain our main result.

Future Work

It may be possible to find enough affinely independent prisms with a weaker constraint on $|E|$, which would improve our main result. It is also possible that another approach might improve our result.

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Another direction would be trying to obtain similar results for other sets of classifiers on subsets of $\mathbb{F}_q^d$. Our classifiers are a themselves a modification of those studied in past work. It is likely that further modifications could lead to even more interesting problems. Finite field VC-dimension problems such as this are a relatively unexplored area, of study so there are many different avenues to explore.

Acknowledgments

Thank you!

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