Virus Dynamics on Star Graphs

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Introduction

Simulation and New Results

Fixed Points and Proofs

Proofs: \( b \leq \frac{1 - a}{\sqrt{n}} \)

Proofs: \( b > \frac{1 - a}{\sqrt{n}} \)
Motivation and Questions

- Epidemiology, networking, and other fields have questions concerning the spread of viruses.

- Using a model for infection and cure rates, look for a steady state or critical threshold relating two rates.

- If there is a steady state, what are the characteristics?

- What other information can we get from this steady state, provided it exists?

- Generalizations?
The Model ($a = 1 - \delta$, $b = \beta$)

A discrete-time **SIS** (Susceptible Infected Susceptible) **model**. Each node is either Susceptible (S) or Infected (I).

Study special graphs: star graphs:

![Star graph](image)

**Figure:** Star graph with 1 central hub and $n$ spokes.
The Model \((a = 1 - \delta, b = \beta)\)

A discrete-time SIS (Susceptible Infected Susceptible) model. Each node is either Susceptible (S) or Infected (I).

Or, a more entertaining view....
The Model \((a = 1 - \delta, b = \beta)\)

A discrete-time SIS (Susceptible Infected Susceptible) model. Each node is either Susceptible (S) or Infected (I).

With infection and cure rates....
A discrete-time SIS (Susceptible Infected Susceptible) model. Each node is either Susceptible (S) or Infected (I).

**Parameters**

- $\beta$: probability at any time step that an infected node infects its neighbors.
- $\delta$: probability at any time step that an infected node is cured.
- $1 - p_{i,t} = (1 - p_{i,t-1}) \zeta_{i,t} + \delta p_{i,t} \zeta_{i,t}$, where $\zeta_{i,t}$ is the probability node $i$ not infected by neighbors at time $t$.
- $\zeta_{i,t} = \prod_{j \sim i} p_{j,t-1} (1 - \beta) + (1 - p_{j,t-1}) = \prod_{j \sim i} 1 - \beta p_{j,t-1}$, where $j \sim i$ means $i$ and $j$ are neighbors (share an edge).
- $1 - p_{i,t} = (1 - p_{i,t-1}) \zeta_{i,t} + \delta p_{i,t} \zeta_{i,t}$, where $\zeta_{i,t}$ is the probability that node $i$ is not infected by its neighbors at time $t$. 
System of Interest

- In limit all spokes behave the same.

- Label hub behavior at time $t$ by $x_t$, spokes by $y_t$. Evolve by

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = F \left( \begin{pmatrix} x_t \\ y_t \end{pmatrix} \right),$$

where

$$F \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} = \begin{pmatrix} 1 - (1 - x)(1 - \beta y)^n - \delta x (1 - \beta y)^n \\ 1 - (1 - y)(1 - \beta x) - \delta y (1 - \beta x) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - (1 - ax)(1 - by)^n \\ 1 - (1 - ay)(1 - bx) \end{pmatrix}.$$
Simulation and New Results
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

Figure: $t = 0$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

**Figure:** $t = 1$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

Figure: $t = 2$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

\[ b \leq \frac{1 - a}{\sqrt{n}} \]

\[ b > \frac{1 - a}{\sqrt{n}} \]

Figure: $t = 3$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

**Figure:** $t = 4$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

Figure: $t = 5$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

**Figure:** $t = 6$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

**Figure:** $t = 7$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = 0.4$, $b = 0.7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

Figure: $t = 8$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

Figure: $t = 9$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

Figure: $t = 10$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

**Figure:** $t = 11$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

**Figure:** $t = 12$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

**Figure:** $t = 13$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

Figure: $t = 14$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

**Figure:** $t = 15$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

**Figure:** $t = 16$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

**Figure:** $t = 17$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = 0.4$, $b = 0.7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

**Figure:** $t = 18$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

Figure: $t = 19$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

Figure: $t = 20$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = 0.4$, $b = 0.7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

**Figure:** $t = 21$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

**Figure:** $t = 22$ (point in upper right needed for display purposes)
Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

**Figure:** $t = 23$ (point in upper right needed for display purposes)
Main Result

**Theorem (BG-TKMS ’13)**

Let $a, b \in (0, 1)$ and $F$ as above.

- For any initial configuration, as time evolves all the spokes converge to a common behavior.

- If $b \leq (1 - a)/\sqrt{n}$ then the virus dies out.

- If $b > (1 - a)/\sqrt{n}$ then all points except $(0, 0)$ evolve to a unique, non-trivial fixed point $(x_f, y_f)$. 
Fixed Points and Proofs

Proofs:

\[ b \leq \frac{1 - a}{\sqrt{n}} \]

Proofs:

\[ b > \frac{1 - a}{\sqrt{n}} \]
Determining Fixed Points of $F$: Partial Fixed Points

Goal is to find fixed points: $F(x, y) = (x, y)$.

Easier: look for partial fixed points:

$$F(x, y) = (x, y') \quad \text{or} \quad F(x, y) = (x', y).$$
Determining Fixed Points of $F$: Partial Fixed Points

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Introduce functions $\phi_1, \phi_2$ so that

- $\forall y \exists y' \text{ st } F(\phi_1(y), y) = (\phi_1(y), y')$.
- $\forall x \exists x' \text{ st } F(x, \phi_2(x)) = (x', \phi_2(x))$.

Can explicitly solve for $\phi_1, \phi_2$.  

Proofs:

- $b \leq (1 - a)/\sqrt{n}$
- $b > (1 - a)/\sqrt{n}$
Determining Fixed Points of $F$: Partial Fixed Curves

Partial fixed points from $\phi_1$ and $\phi_2$ when (from left to right) $b < \frac{1-a}{\sqrt{n}}$, $b = \frac{1-a}{\sqrt{n}}$, $b > \frac{1-a}{\sqrt{n}}$ ($b = 3$, $n = 4$, $a = .1, .4, .7$).

$$\phi_1(y) = \frac{1 - (1 - by)^n}{1 - a(1 - by)^n} \quad \phi_2(x) = \frac{bx}{1 - a + abx}.$$
Determining Fixed Points of $F$: Regions: $b > (1 - a)/\sqrt{n}$

**Figure:** The four regions determined by the partial fixed point functions when $b > (1 - a)/\sqrt{n}$.

Analysis easy if $b \leq (1 - a)/\sqrt{n}$; $(0, 0)$ only fixed point.

Proof unique additional fixed point when $b > (1 - a)/\sqrt{n}$: concavity of the partial fixed point curves and value of derivatives at origin.
Proofs: $b \leq (1 - a)/\sqrt{n}$
Convergence Case \( b \leq \frac{(1-a)}{\sqrt{n}} \)

**Theorem**

Assume \( b < \frac{(1-a)}{\sqrt{n}} \). Then iterates of any point under \( F \) converge to the trivial fixed point \((0, 0)\).

Proved with MVT and an eigenvalue analysis of the resulting matrix.

**Lemma:** Let \( a, b \in (0, 1) \) with \( b < \frac{(1-a)}{\sqrt{n}} \), and let \( \lambda_1 \geq \lambda_2 \) denote the eigenvalues of the matrix

\[
\begin{pmatrix}
    a\alpha & nb\beta \\
    b\gamma & a\delta
\end{pmatrix},
\]

where \( \alpha, \beta, \gamma, \delta \in [0, 1] \). Then

\(-1 < \lambda_1, \lambda_2 < 1\).
(0, 0) is the unique fixed point.
Proof

- \((0, 0)\) is the unique fixed point.

- \(c(t) = (1 - t) \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} x \\ y \end{pmatrix}\), \(c'(t) = \begin{pmatrix} x \\ y \end{pmatrix}\), the line connecting the trivial fixed point to \(\begin{pmatrix} x \\ y \end{pmatrix}\), with \(c(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\) and \(c(1) = \begin{pmatrix} x \\ y \end{pmatrix}\).


Proof

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- \(c(t) = (1 - t) \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} x \\ y \end{pmatrix}, \quad c'(t) = \begin{pmatrix} x \\ y \end{pmatrix}\), the line connecting the trivial fixed point to \(\begin{pmatrix} x \\ y \end{pmatrix}\), with \(c(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\) and \(c(1) = \begin{pmatrix} x \\ y \end{pmatrix}\).

- \(F(t) = f(c(t)) = \begin{pmatrix} 1 - (1 - atx)(1 - bty)^n \\ 1 - (1 - aty)(1 - btx) \end{pmatrix}\).
Proof

- $(0, 0)$ is the unique fixed point.

- \( c(t) = (1 - t) \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} x \\ y \end{pmatrix} \), \( c'(t) = \begin{pmatrix} x \\ y \end{pmatrix} \), the line connecting the trivial fixed point to \( \begin{pmatrix} x \\ y \end{pmatrix} \), with \( c(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) and \( c(1) = \begin{pmatrix} x \\ y \end{pmatrix} \).

- \( F(t) = f(c(t)) = \begin{pmatrix} 1 - (1 - atx)(1 - bty)^n \\ 1 - (1 - aty)(1 - btx) \end{pmatrix} \).

- \( F'(t) = \left( \begin{array}{cc} a(1 - bty)^n & nb(1 - atx)(1 - bty)^{n-1} \\ b(1 - aty) & a(1 - btxu) \end{array} \right) \begin{pmatrix} x \\ y \end{pmatrix}. \)
Proof (continued)

Apply the one-dimensional chain rule twice, once to the $x$-coordinate function and once to the $y$-coordinate function.
Proof (continued)

- Apply the one-dimensional chain rule twice, once to the $x$-coordinate function and once to the $y$-coordinate function.

- Get $t_1$ and $t_2$ such that

$$f \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) - f \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) =$$

$$\begin{pmatrix} a(1 - bt_1 y)^n & nb(1 - at_1 x)(1 - bt_1 y)^{n-1} \\ b(1 - at_2 y) & a(1 - bt_2 x) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. $$

Proofs:

- $b \leq (1 - a)/\sqrt{n}$
- $b > (1 - a)/\sqrt{n}$
Proof (continued)

- Apply the one-dimensional chain rule twice, once to the $x$-coordinate function and once to the $y$-coordinate function.

- Get $t_1$ and $t_2$ such that

$$f \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) - f \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} a(1 - bt_1 y)^n & nb(1 - at_1 x)(1 - bt_1 y)^{n-1} \\ b(1 - at_2 y) & a(1 - bt_2 x) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

- We have a contraction map on a compact space, completing the proof.
Proofs: $b > \frac{(1 - a)}{\sqrt{n}}$
Preliminary Results

Recall

**Figure:** The four regions determined by the partial fixed point functions when $b > \left(1 - a\right)/\sqrt{n}$.
Preliminary Results

Key lemmas (proofs by algebra):

- Points in Region I strictly increase in $x$ and $y$ on iteration by $F$, and points in Region III strictly decrease in $x$ and $y$ on iteration.

- Points in Region I iterate inside Region I under $F$, and points in Region III iterate inside Region III under $F$.

- All non-trivial points in Regions I and III converge to the non-trivial fixed point under $F$.

Armed with the above lemmas, we now complete the proof.
Proof of Limiting Behavior

- Consider any rectangle in $[0, 1]^2$ whose lower left vertex is not $(0, 0)$.  

Proofs: $b \leq (1 - a)/\sqrt{n}$

Proofs: $b > (1 - a)/\sqrt{n}$
Proof of Limiting Behavior

- Consider any rectangle in $[0, 1]^2$ whose lower left vertex is not $(0, 0)$.

- Assume the lower left and upper right vertices are in Regions I and III.
Proof of Limiting Behavior

- Consider any rectangle in $[0, 1]^2$ whose lower left vertex is not $(0, 0)$.

- Assume the lower left and upper right vertices are in Regions I and III.

- Image of rectangle under $F$ is strictly contained in rectangle (image of the lower left (respectively, upper right) point has both coordinates smaller (respectively, larger) than any other iterate).
Proof of Limiting Behavior

- Consider any rectangle in $[0, 1]^2$ whose lower left vertex is not $(0, 0)$.

- Assume the lower left and upper right vertices are in Regions I and III.

- Image of rectangle under $F$ is strictly contained in rectangle (image of the lower left (respectively, upper right) point has both coordinates smaller (respectively, larger) than any other iterate).

- As lower left and upper right vertices iterate to the non-trivial fixed points (in Regions I and III), so too do all the other points in rectangle.
Behavior Conjectures

Corollary

The amount of time it takes for all points to converge is the maximum of the time it takes \( \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) to converge, for \( \epsilon_1, \epsilon_2 \to 0 \).

Conjecture

Points in Region II and IV exhibit one of two behaviors, dependent on \( a, b, n \). Either:

1. All points in Region II iterate outside Region II and all points in Region IV iterate outside Region IV ("flipping behavior"), or
2. All points in Region II iterate outside Region IV and all points in Region IV iterate outside Region II
Conclusions and References

- Can extend to Generalized Star Graphs.

  