Virus Dynamics on Star Graphs: A Generalized Model

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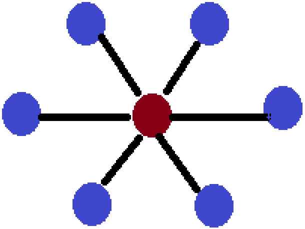
Motivation for the Problem

- Epidemiology, networking, and other fields have questions concerning the spread of viruses.
- Using a model for infection and cure rates, look for a steady state, or critical threshold relating two rates.
- If there is a steady state, what are the characteristics?
- What other information can we get from this steady state, provided it exists?
- Generalizations?

The Model

A discrete-time SIS (Susceptible Infected Susceptible) model. Each node is either Susceptible (S) or Infected (I).



The Model

A discrete-time SIS (Susceptible Infected Susceptible) model. Each node is either Susceptible (S) or Infected (I).

Parameters

- β is the probability at any time step that an infected node infects its neighbors.
- δ is the probability at any time step that an infected node is cured.
- $1 - p_{i,t} = (1 - p_{i,t-1})\,\zeta_{i,t} + \delta p_{i,t}\zeta_{i,t}$, where $\zeta_{i,t}$ is the probability node i not infected by neighbors at time t.
- $\zeta_{i,t} = \prod_{j\sim i} p_{j,t-1}\,(1-\beta) + (1 - p_{j,t-1}) = \prod_{j\sim i} 1 - \beta p_{j,t-1}$, where $j \sim i$ says i and j are neighbors, connected by edge of the graph.
- $1 - p_{i,t} = (1 - p_{i,t-1})\,\zeta_{i,t} + \delta p_{i,t}\zeta_{i,t}$, where $\zeta_{i,t}$ is the probability that node i is not infected by its neighbors at time t.

A Different View of the Model

- Let's consider specific graph topology of a star graph, then we can alter the model to one of "hubs" and "spokes"
- Suppose a graph has $n+1$ nodes, the hub is numbered 0 and the spokes are numbered 1 through n.

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = F\begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

where

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} 1-(1-x)(1-\beta y)^n - \delta x(1-\beta y)^n \\ 1-(1-y)(1-\beta x) - \delta y(1-\beta x) \end{pmatrix}$$
$$= \begin{pmatrix} 1-(1-ax)(1-by)^n \\ 1-(1-ay)(1-bx) \end{pmatrix};$$

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

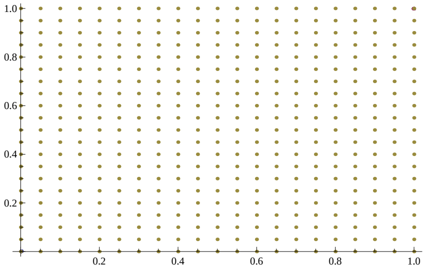


Figure: $t = 0$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

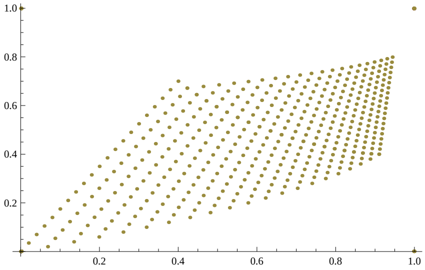


Figure: $t = 1$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

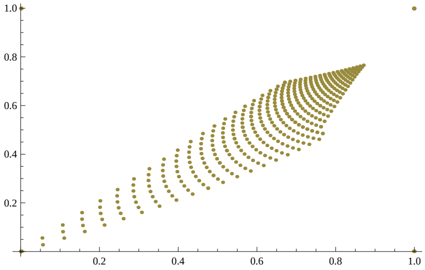


Figure: $t = 2$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

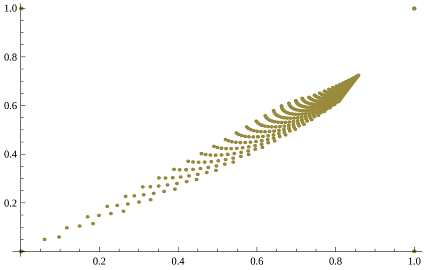


Figure: $t = 3$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

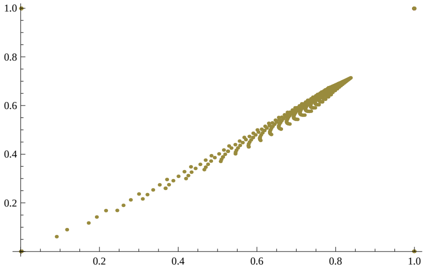


Figure: $t = 4$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

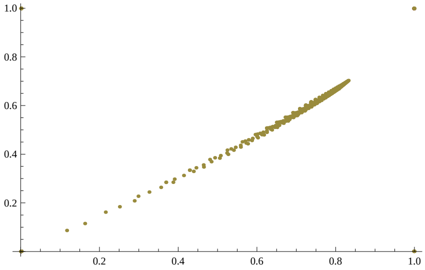


Figure: $t = 5$ (point in upper right needed for display purposes)

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

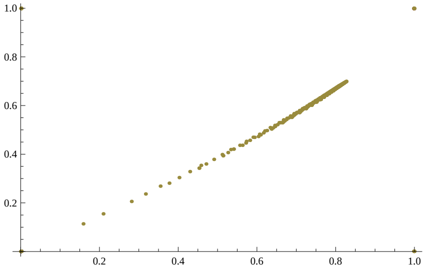


Figure: $t = 6$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

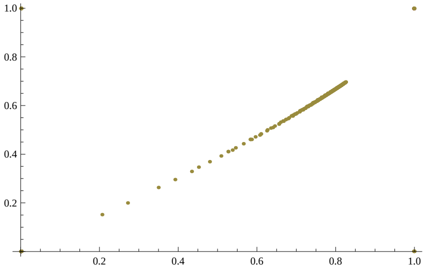


Figure: $t = 7$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

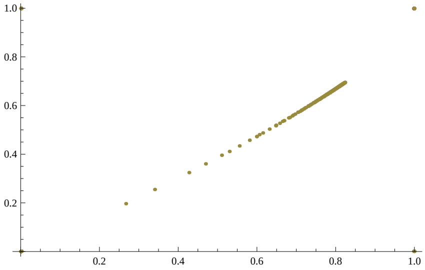


Figure: $t = 8$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

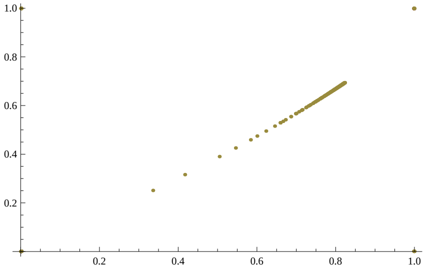


Figure: $t = 9$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

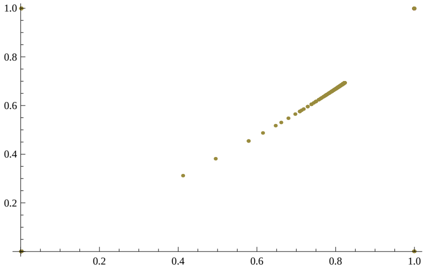


Figure: $t = 10$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

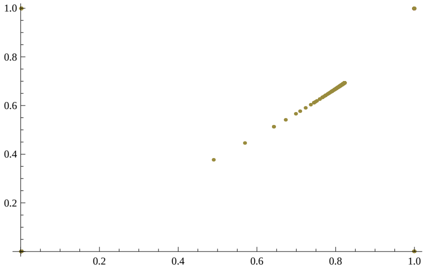


Figure: $t = 11$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

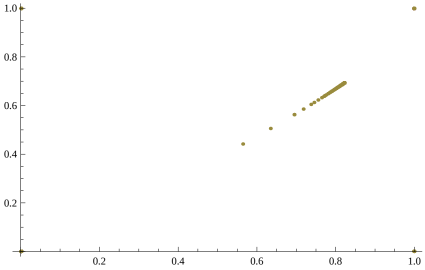


Figure: $t = 12$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

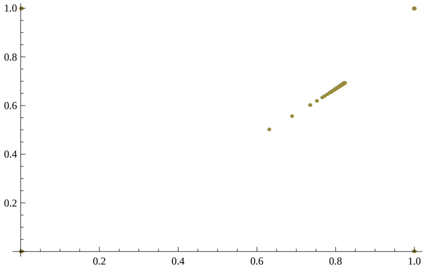


Figure: $t = 13$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

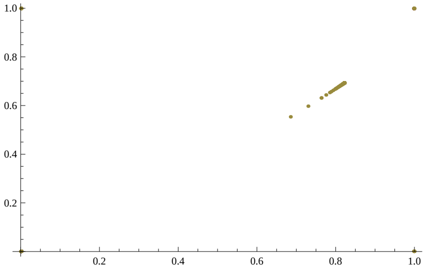


Figure: $t = 14$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

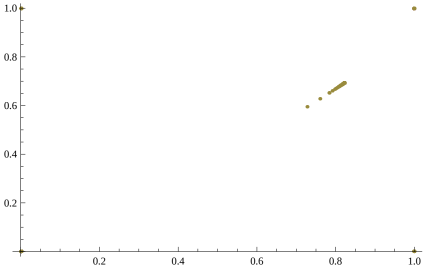


Figure: $t = 15$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

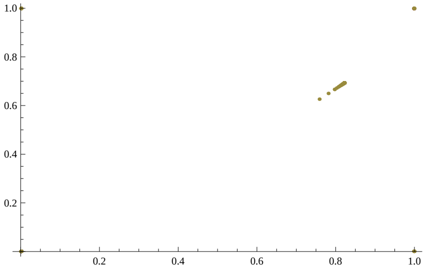


Figure: $t = 16$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

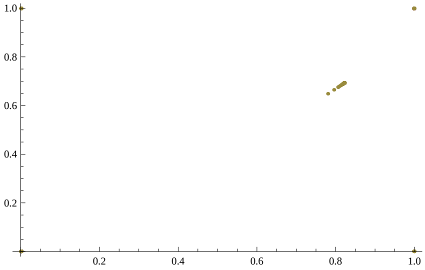


Figure: $t = 17$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

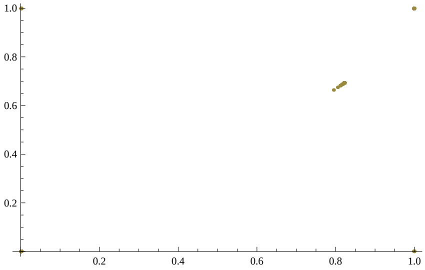


Figure: $t = 18$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

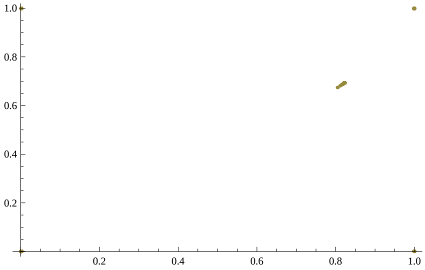


Figure: $t = 19$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

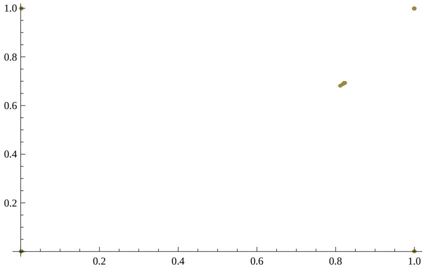


Figure: $t = 20$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

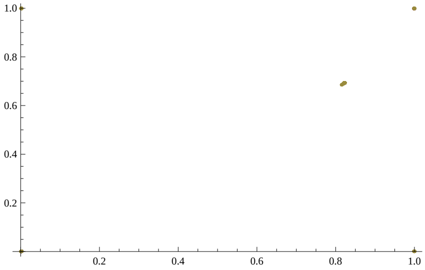


Figure: $t = 21$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

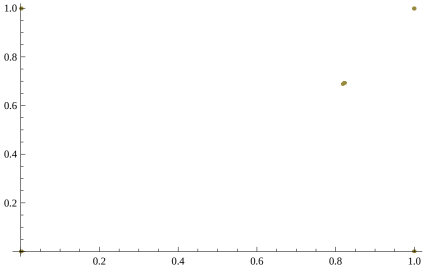


Figure: $t = 22$ (point in upper right needed for display purposes)

Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

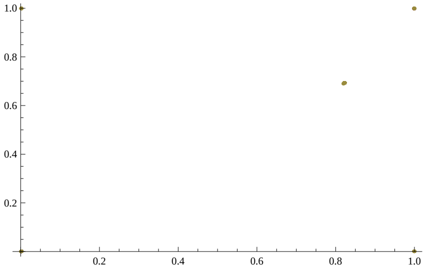


Figure: $t = 23$ (point in upper right needed for display purposes)

Our Main Result

Theorem

Let $a, b \in (0,1)$ *and* F *as above.*

- *For any initial configuration, as time evolves all the spokes converge to a common behavior.*
- *If* $b \leq (1-a)/\sqrt{n}$ *then the virus dies out.*
- *If* $b > (1-a)/\sqrt{n}$ *then all points except* $(0,0)$ *evolves to a unique, non-trivial fixed point* (x_f, y_f)*.*

Determining Fixed Points of F

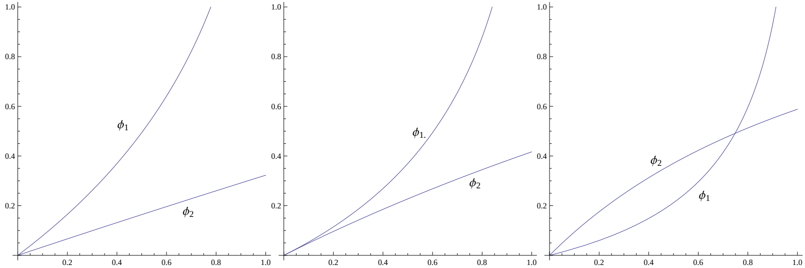


Figure: Partial fixed points from ϕ_1 and ϕ_2 when (from left to right) $b < \frac{1-a}{\sqrt{n}}$, $b = \frac{1-a}{\sqrt{n}}$, $b > \frac{1-a}{\sqrt{n}}$ ($b = 3, n = 4, a = .1, .4, .7$).

$$\begin{aligned} \phi_1(y) &= \frac{1-(1-by)^n}{1-a(1-by)^n} \\ \phi_2(x) &= \frac{bx}{1-a+abx}. \end{aligned} \tag{1}$$

Determining Fixed Points of F

Theorem

Consider the map F, and assume $b < \frac{1-a}{\sqrt{n}}$. Then the trivial fixed point is the unique fixed point in $[0,1]^2$.

Theorem

For the map F, when $b > \frac{1-a}{\sqrt{n}}$ there exists a unique non-trivial fixed point in $[0,1]^2$.

Convergence Case $b \leq \frac{(1-a)}{\sqrt{n}}$

Theorem

Assume $b < (1-a)/\sqrt{n}$*. Then iterates of any point under* F *converge to the trivial fixed point* $(0,0)$*.*

Proved this with the Mean Value Theorem and an eigenvalue analysis of the resulting matrix.

Convergence: Case $b > \frac{(1-a)}{\sqrt{n}}$

Lemma

Points in Region I strictly increase in x and y on iteration by F, and points in Region III strictly decrease in x and y on iteration.

The proof for points in Region III is exactly analogous except with the inequalities flipped. Thus

$$x > \frac{1-(1-by)^n}{1-a(1-by)^n} \qquad (2)$$

and

$$y > \frac{bx}{1-a+abx} \qquad (3)$$

imply that

$$x > 1-(1-ax)(1-by)^n = f_1(x); \quad y > 1-(1-ay)(1-bx) = f_2(y).$$

Lemma

Points in Region I iterate inside Region I under F, and points in Region III iterate inside Region III under F.

We prove that for a point $\begin{pmatrix} x \\ y \end{pmatrix}$ in Region I, its iterated x-coordinate satisfies (2) and its iterated y-coordinate satisfies (3). We do this using algebra as well.

Lemma

All non-trivial points in Regions I and III converge to the non-trivial fixed point under F.

Armed with the above lemmas, we now complete the proof of Theorem 0.1.

Theorem

Any non-trivial point in $[0, 1]^2$ *converges to the unique non-trivial fixed point under* F.

Corollary

The amount of time it takes for all points to converge is the maximum of the time it takes $\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *to converge, for* $\epsilon_1, \epsilon_2 \to 0$.

Conjecture

Points in Region II and IV exhibit one of two behaviors, dependent on a, b, n. *Either:*

1. *All points in Region II iterate outside Region II and all points in Region IV iterate outside Region IV ("flipping behavior"), or*
2. *All points in Region II iterate outside Region IV and all points in Region IV iterate outside Region II*

Conclusions

- Generalized Star Graphs
- Conclusions