Analyzing Virus Dynamics on k-level Starlike Graphs

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Introduction

Motivation and Questions: Virus propagation on starlike graphs

- Epidemiology, networking, and other fields have questions concerning the spread of viruses.
- Using a model for infection and cure rates, look for a steady state or critical threshold relating two rates.
- If there is a steady state, what are the characteristics?
- What other information can we get from this steady state, provided it exists?
- Generalizations?

The Model ($a = 1 - \delta$, $b = \beta$)

A discrete-time **SIS** (**S**usceptible **I**nfected **S**usceptible) **model**. Each node is either **S**usceptible (**S**) or Infected (**I**).

Study special graphs: starlike graphs:

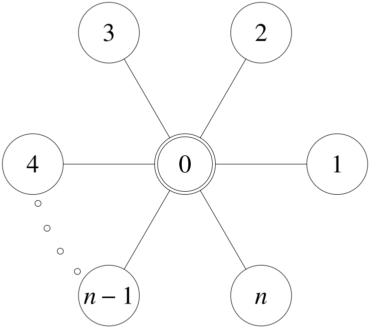

Figure: Starlike graph with 1 central hub and n spokes.

The Model ($a = 1-\delta$, $b = \beta$)

A discrete-time **SIS** (**S**usceptible **I**nfected **S**usceptible) **model**. Each node is either **S**usceptible (**S**) or Infected (**I**).

Parameters

- $p_{i,t}$: probability that node i is infected at time t.
- $\beta = b$: probability at any time step that an infected node infects its neighbors.
- $\delta = 1-a$: probability at any time step that an infected node is cured.
- $1 - p_{i,t} = (1 - p_{i,t-1})\,\zeta_{i,t} + \delta p_{i,t}\zeta_{i,t}$, where $\zeta_{i,t}$ is the probability that node i is not infected by its neighbors at time t.
- $\zeta_{i,t} = \prod_{j\sim i} p_{j,t-1}(1-\beta) + (1-p_{j,t-1}) = \prod_{j\sim i} 1 - \beta p_{j,t-1}$, where $j \sim i$ means i and j are neighbors (share an edge).

Previous Work: 2-level Starlike Graphs

- In limit all spokes behave the same.
- $\beta = b$: probability an infected node infects neighbors.
- $\delta = 1 - a$: probability an infected node is cured.
- Label hub behavior at time t by x_t, spokes by y_t. Evolve by

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = F \begin{pmatrix} x_t \\ y_t \end{pmatrix},$$

where

$$\begin{aligned} F\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} 1-(1-x)(1-\beta y)^n - \delta x(1-\beta y)^n \\ 1-(1-y)(1-\beta x) - \delta y(1-\beta x) \end{pmatrix} \\ &= \begin{pmatrix} 1-(1-ax)(1-by)^n \\ 1-(1-ay)(1-bx) \end{pmatrix}. \end{aligned}$$

Previous Work: 2-level Starlike Graphs

Theorem (BG-TKMS '13)

Let $a, b \in (0, 1)$ and F as above.

- *For any initial configuration, as time evolves all the spokes converge to a common behavior.*
- *If $b \leq (1-a)/\sqrt{n}$ then the virus dies out.*
- *If $b > (1-a)/\sqrt{n}$ then all points except $(0, 0)$ evolve to a unique, non-trivial fixed point (x_f, y_f).*

Viruslike dynamics on starlike graphs (Thealexa Becker, Alec Greaves-Tunnell, Leo Kontorovich, Steven J. Miller and Karen Shen), the Journal of Nonlinear Systems and Applications **4** (2013), no. 1, 53–63.

Our Work

- Can this model be extended to 3-level? (1 central hub connected to n_1 spokes, each of which are connected to n_2 spokes)
- What about an arbitrary number of levels? (k-level)
- Approach: Start with 3-level, extend to k-level.

3-level System

- In limit all 2-level spokes behave the same, and all 3-level spokes behave the same.
- Label hub behavior at time t by x_t, 2-level spokes by y_t, 3-level spokes by z_t. Evolve by

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = F \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix},$$

where

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-(1-x)(1-\beta y)^{n_1} - \delta x (1-\beta y)^{n_1} \\ 1-(1-y)(1-bx)(1-bz)^{n_2} - \delta y(1-bx)(1-bz)^{n_2} \\ 1-(1-z)(1-\beta y) - \delta z (1-\beta y) \end{pmatrix}$$

$$= \begin{pmatrix} 1-(1-ax)(1-by)^{n_1} \\ 1-(1-ay)(1-bx)(1-bz)^{n_2} \\ 1-(1-az)(1-by) \end{pmatrix}.$$

Main Result

Theorem (Steven J. Miller, Akihiro Takigawa)

Let $a, b \in (0, 1)$ *and* F *as above.*

- *For any initial configuration, as time evolves all the spokes converge to a common behavior.*
- *If* $b \leq (1-a)/\sqrt{n_1+n_2}$ *then the virus dies out.*
- *If* $b > (1-a)/\sqrt{n_1+n_2}$ *then all points except* $(0, 0, 0)$ *evolve to a unique, non-trivial fixed point* (x_f, y_f, z_f).

Fixed Points and Proofs

Determining Fixed Points of F: Introduction

2-level case:
Goal is to find fixed points: $F(x, y) = (x, y)$.

Easier: look for partial fixed points:

$$F(x, y) = (x, y') \quad \text{or} \quad F(x, y) = (x', y).$$

Determining Fixed Points of F: Introduction

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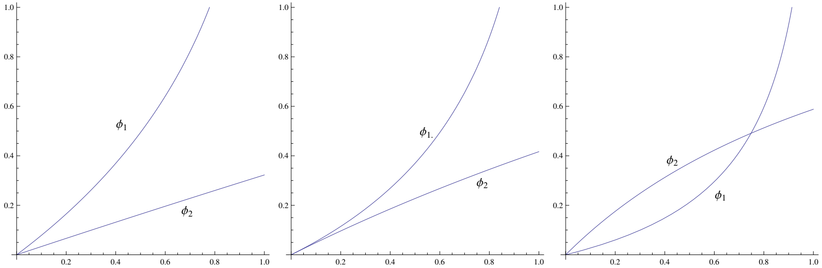
Introduce functions ϕ_1, ϕ_2 so that

- $\forall y \; \exists y'$ st $F(\phi_1(y), y) = (\phi_1(y), y')$.
- $\forall x \; \exists x'$ st $F(x, \phi_2(x)) = (x', \phi_2(x))$.

Can explicitly solve for ϕ_1, ϕ_2.

Determining Fixed Points of F: Introduction



Partial fixed points from ϕ_1 and ϕ_2 when (from left to right) $b < \frac{1-a}{\sqrt{n}}$, $b = \frac{1-a}{\sqrt{n}}$, $b > \frac{1-a}{\sqrt{n}}$ ($b = .3$, $n = 4$, $a = .1, .4, .7$).

$$\phi_1(y) = \frac{1-(1-by)^n}{1-a(1-by)^n} \quad \phi_2(x) = \frac{bx}{1-a+abx}.$$

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

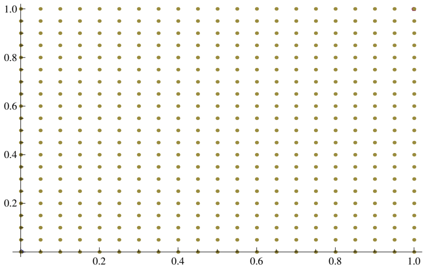


Figure: $t = 0$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

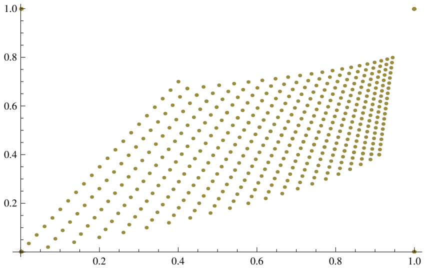


Figure: $t = 1$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

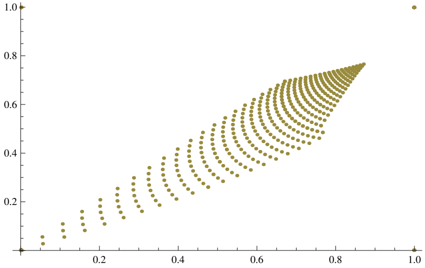


Figure: $t = 2$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

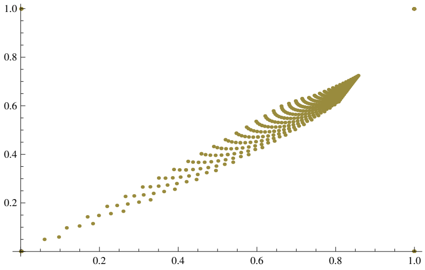


Figure: $t = 3$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

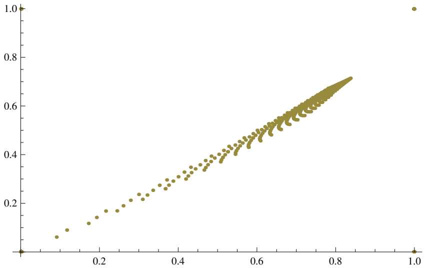


Figure: $t = 4$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

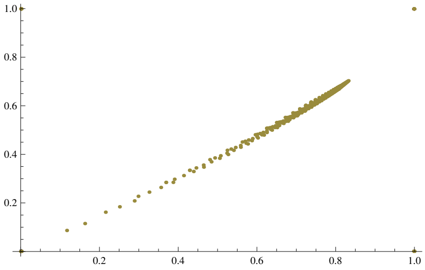


Figure: $t = 5$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

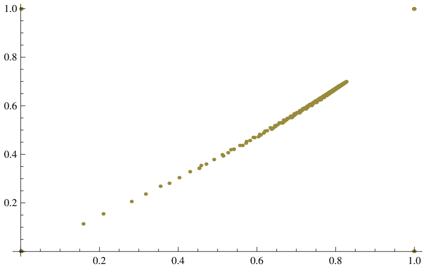


Figure: $t = 6$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

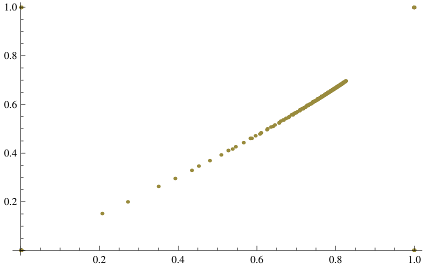


Figure: $t = 7$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

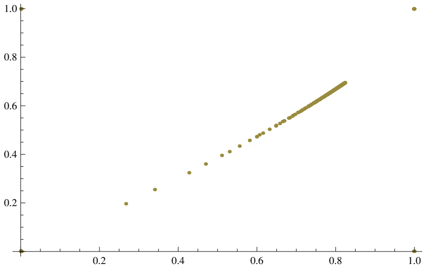


Figure: $t = 8$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

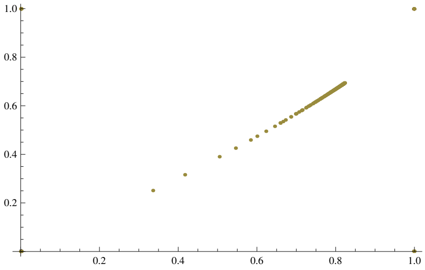


Figure: $t = 9$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

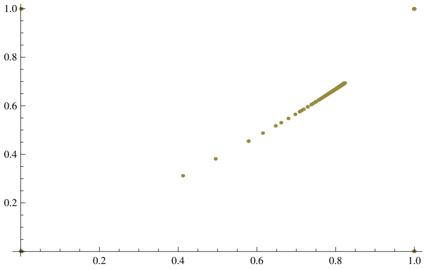


Figure: $t = 10$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1 - a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

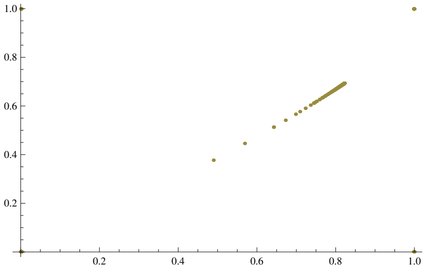


Figure: $t = 11$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

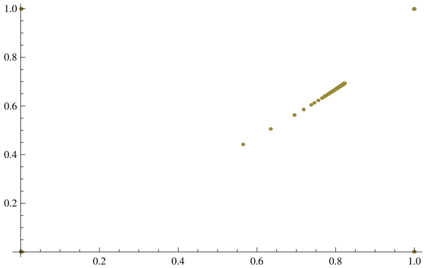


Figure: $t = 12$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

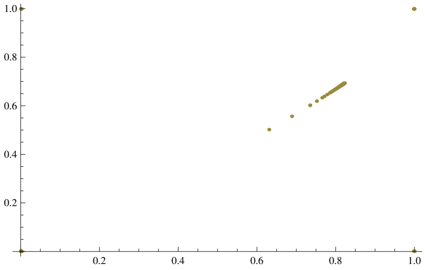


Figure: $t = 13$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

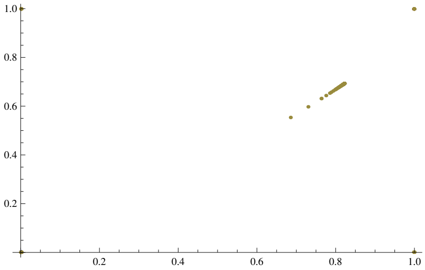


Figure: $t = 14$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1 - a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

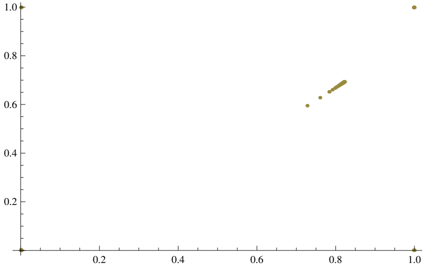


Figure: $t = 15$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

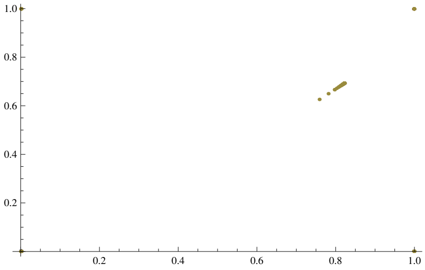


Figure: $t = 16$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

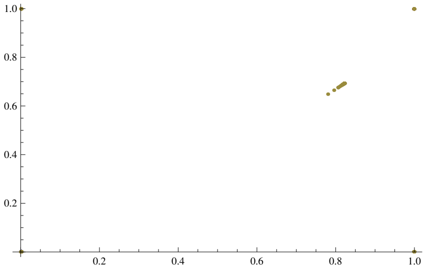


Figure: $t = 17$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

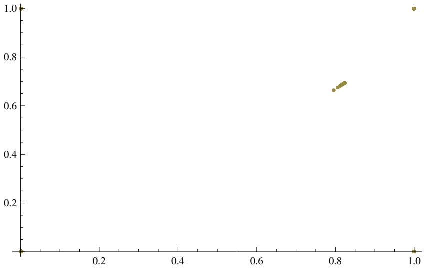


Figure: $t = 18$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

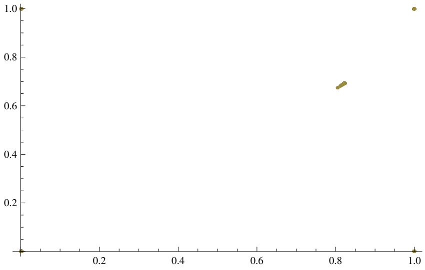


Figure: $t = 19$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

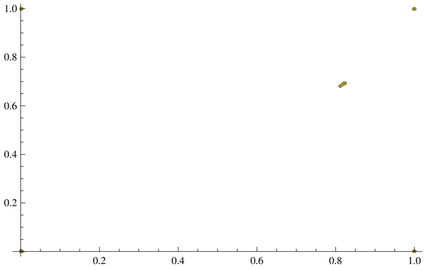


Figure: $t = 20$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

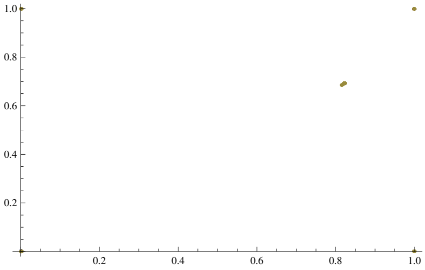


Figure: $t = 21$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

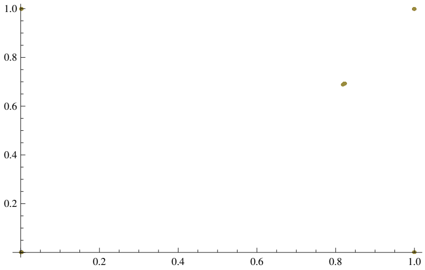


Figure: $t = 22$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

b: probability infected node infects neighbors, $1-a$ probability infected node not cured; below $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

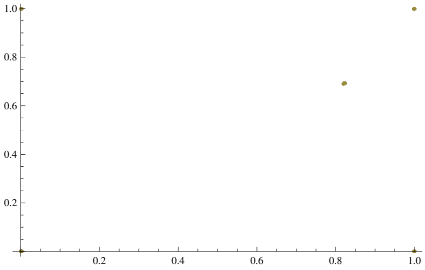


Figure: $t = 23$ (point in upper right needed for display purposes)

Determining Fixed Points of F: Introduction

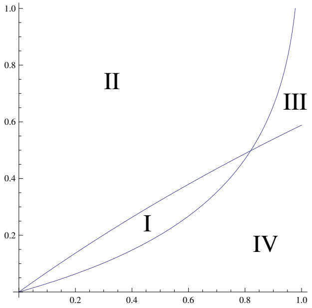


Figure: The four regions determined by the partial fixed point functions when $b > (1-a)/\sqrt{n}$.

Analysis easy if $b \leq (1-a)/\sqrt{n}$; $(0,0)$ only fixed point.

Proof unique additional fixed point when $b > (1-a)/\sqrt{n}$: concavity of the partial fixed point curves and value of derivatives at origin.

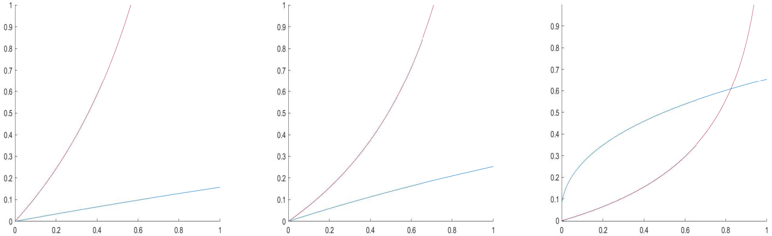
Determining Fixed Points of F: Partial Fixed Points

For 3-level, goal is to find fixed points:
$F(x, y, z) = (x, y, z)$.

Follow similar steps as before:

- Look at partial fixed points, define functions ϕ_1, ϕ_2, ϕ_3.
- New! Take the intersection of ϕ_1 with ϕ_3, and ϕ_2 with ϕ_3 to reduce the 3-dimensional problem to a 2-dimensional one.

Determining Fixed Points of F: Partial Fixed Curves



Red is $\phi_1 \circ \phi_3$, blue is $\phi_2 \circ \phi_3$.

Partial fixed points from $\phi_1 \circ \phi_3$ and $\phi_2 \circ \phi_3$ when (from left to right) $b < \frac{1-a}{\sqrt{n_1+n_2}}$, $b = \frac{1-a}{\sqrt{n_1+n_2}}$, $b > \frac{1-a}{\sqrt{n_1+n_2}}$

($b = 0.08, 0.125, 0.4$, $n_1 = 6$, $n_2 = 10$, $a = 0.5$).

(Note: In the rightmost plot, $\phi_2 \circ \phi_3$ contains $(0, 0)$, but it is disconnected)

Determining Fixed Points of F: Locations of fixed points

Using convexity / concavity of the partial fixed point curves:

If $b \leq (1-a)/\sqrt{n_1+n_2}$, then $(0,0,0)$ is the only fixed point since $\phi_1 \circ \phi_3$ is convex, $\phi_2 \circ \phi_3$ is concave, and slope of $\phi_1 \circ \phi_3$ is greater than slope of $\phi_2 \circ \phi_3$ at the origin.

Similarly, proved unique additional fixed point when $b > (1-a)/\sqrt{n_1+n_2}$.

Proofs: $b \leq (1-a)/\sqrt{n_1+n_2}$

Convergence Case $b \le \frac{(1-a)}{\sqrt{n_1+n_2}}$

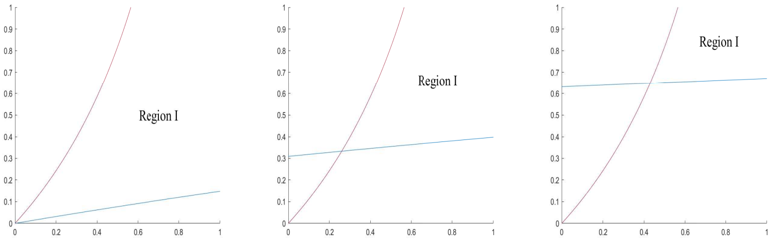
Theorem (Steven J. Miller, Akihiro Takigawa)

Assume $b \le (1-a)/\sqrt{n_1+n_2}$. Then iterates of any point under F converge to the trivial fixed point $(0,0,0)$.

Outline of our argument:

- Proved by first focusing on the limiting behavior of points inside the region $x > \phi_1(y,z)$, $y > \phi_2(x,z)$, $z > \phi_3(x,y)$. (Hereby called Region I for brevity)
- Then, consider a cuboid with vertices in Region I.
- Finally, use squeeze theorem to show that any point in the cuboid exhibits the same limiting behavior.

$b \leq \frac{(1-a)}{\sqrt{n_1+n_2}}$: Visualization of Region I



Red is ϕ_1, blue is ϕ_2.
When $b \leq \frac{1-a}{\sqrt{n_1+n_2}}$, slices of Region I on the xy-plane at (from left to right) $z = 0$, $z = 0.25$, $z = 0.75$.
($b = 0.08, n_1 = 6, n_2 = 10, a = 0.5$).

Key lemmas (proofs by algebra):

- Points in Region I strictly decrease in x, y and z on iteration by F.
- Points in Region I iterate inside Region I under F.
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Armed with the above lemmas, we now complete the proof.

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- As $(0,0,0)$ iterates to $(0,0,0)$ by F, and (x_u, y_u, z_u) iterates to $(0,0,0)$ by F, so do any point in the cuboid.

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- As $(0,0,0)$ iterates to $(0,0,0)$ by F, and (x_u, y_u, z_u) iterates to $(0,0,0)$ by F, so do any point in the cuboid.
- Note: We can take larger and larger cuboids to encompass all non-trivial points in $[0,1]^3$.

Extension to k-level

Comparison of 3-level to k-level

Recall that for 3-level:

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-(1-ax)(1-by)^{n_1} \\ 1-(1-ay)(1-bx)(1-bz)^{n_2} \\ 1-(1-az)(1-by) \end{pmatrix}.$$

Now, k-level:

$$F\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_k \end{pmatrix} = \begin{pmatrix} 1-(1-ad_1)(1-bd_2)^{n_1} \\ 1-(1-ad_2)(1-bd_1)(1-bd_3)^{n_2} \\ 1-(1-ad_3)(1-bd_2)(1-bd_4)^{n_3} \\ \vdots \\ 1-(1-ad_k)(1-bd_{k-1}) \end{pmatrix}$$

($x, y, z, \dots$ relabeled as $d_1, d_2, d_3, \dots$ for simplicity)
Very similar!

Determining Fixed Points of F: Partial Fixed Points

Once again, goal is to find fixed points:
$F(d_1, \ldots, d_k) = (d_1, \ldots, d_k)$.

Follow the same steps as before:

- Look at partial fixed points, define functions $\phi_1, \ldots, \phi_k$.
- Take the intersection of ϕ_1 with $\phi_3, \ldots, \phi_k$, and ϕ_2 with $\phi_3, \ldots, \phi_k$ to reduce the k-dimensional problem to a 2-dimensional one.

Complications are introduced by the composition of an arbitrary number of ϕ's.

Determining Fixed Points of F: Partial Fixed Points

Key lemmas (proved through algebra):

- $\phi_1(d_2, \ldots, d_k)$ is convex.
- $\phi_k(d_1, \ldots, d_{k-1})$ is concave.
- For all levels $2 \leq m \leq k-1$, $\phi_m(d_1, \ldots, d_{m-1}, \ldots, d_k)$ is non-decreasing in each argument, and is concave.
- The composition of a concave function $f : [0,1]^2 \to [0,1]$ that is non-decreasing in each argument and a concave function $g : [0,1] \to [0,1]$ is concave.
  $\implies$ the composition of $\phi_2, \ldots, \phi_k$ is non-decreasing in each argument and is concave.

Determining Fixed Points of F: k-level

To complete our analysis,

- We have ϕ_1 is convex.
- $(0, \dots, 0)$ is always a fixed point as ϕ_2 passes through $(0, 0)$, and every ϕ_m returns 0 for some argument.
- The composition of $\phi_2, \dots, \phi_k$ is concave, and non-decreasing.
  - Partial fixed point curves display same behavior as 3-level!
- We appeal to our concavity argument from 3-level to determine that when $b > (1-a)/\sqrt{n_1 + \cdots + n_{k-1}}$, there is always one non-trivial fixed point.

k-level: Limiting Behavior

- Just as in 3-level, k-level always has a trivial fixed point, and when $b > (1-a)/\sqrt{n_1 + \cdots + n_{k-1}}$, one additional non-trivial fixed point.

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k-level: Limiting Behavior

- Just as in 3-level, k-level always has a trivial fixed point, and when $b > (1-a)/\sqrt{n_1 + \cdots + n_{k-1}}$, one additional non-trivial fixed point.
- Is limiting behavior similar in k-level as well?
- Yes! Proved by using similar methods to 3-level. Use a k-orthotope instead of a cuboid.

Main Result: b: probability infected node infects, $1-a$ probability infected not cured

Theorem (Steven J. Miller, Akihiro Takigawa)

Let $a, b \in (0,1)$ and F as above.

- *For any initial configuration, as time evolves all the spokes on the same level converge to a common behavior.*
- *If $b \leq (1-a)/\sqrt{n_1+n_2+\cdots+n_{k-1}}$ then the virus dies out.*
- *If $b > (1-a)/\sqrt{n_1+n_2+\cdots+n_{k-1}}$ then all points except $(0,\dots,0)$ evolve to a unique, non-trivial fixed point $(d_{1f},\dots,d_{kf})$.*

Future Work

- The current model is good for virus propagation behavior in regions where there is one large population hub, and numerous adjacent areas dependent on it.
  - Examples: NYC + tri-state area, London + Greater London area

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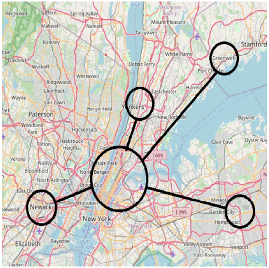

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  - Examples: Northeast corridor (Boston+NYC+Philadelphia+Washington D.C.), Japan (Tokyo+Nagoya+Osaka)

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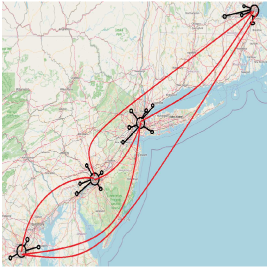
- What about regions with multiple large population hubs?
  - Examples: Northeast corridor (Boston+NYC+Philadelphia+Washington D.C.), Japan (Tokyo+Nagoya+Osaka)
- Consider a complete graph (each node is connected to every other node), but each node expands to a k-level starlike graph.



Future Work

- How does the number of nodes affect the probability of infection?
  - Does increasing the number of nodes increase the probability of infection?
  - Are nodes on one level more influential than other levels in terms of the effect on the probability of infection?
- Rate of convergence to fixed points?

Conclusions and References

- Can extend to Generalized Star Graphs.
  - 2-level to 3-level requires a bit of work, but from 3-level to k-level is straightforward.
- Thealexa Becker, Alec Greaves-Tunnell, Leo Kontorovich, Steven J. Miller and Karen Shen), *Virus Dynamics on Spoke and Star Graphs*, the Journal of Nonlinear Systems and Applications **4** (2013), no. 1, 53–63. http://arxiv.org/pdf/1111.0531.

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Many thanks to the organizers for the invitation.

Appendix

Appendix: Determining 3-level Partial Fixed Points

Goal: find fixed points $F(x, y, z) = (x, y, z)$.

Start by looking for partial fixed points:
$F(x, y, z) = (x, y', z')$ or $F(x, y, z) = (x', y, z')$ or
$F(x, y, z) = (x', y', z)$

Introduce functions ϕ_1, ϕ_2, ϕ_3 so that

- $\forall y, z \; \exists y', z'$ s.t. $F(\phi_1(y, z), y, z) = (\phi_1(y, z), y', z')$.
- $\forall x, z \; \exists x', z'$ s.t. $F(x, \phi_2(x), z) = (x', \phi_2(x), z')$.
- $\forall x, y \; \exists x', y'$ st $F(x, y, \phi_3(x, y)) = (x', y', \phi_3(x, y))$.

Can explicitly solve for tractable ϕ_1, ϕ_2, ϕ_3.

Appendix: Determining 3-level Partial Fixed Points

Solve:

- $x = f_1(x, y, z)$
- $y = f_2(x, y, z)$
- $z = f_3(x, y, z)$

We get:

- $\phi_1(y, z) = \frac{1-(1-by)^{n_1}}{1-a(1-by)^{n_1}}$ (x-coordinate is unchanged on iteration)
- $\phi_2(x, z) = \frac{1-(1-bx)(1-bz)^{n_2}}{1-a(1-bx)(1-bz)^{n_2}}$ (y-coordinate is unchanged on iteration)
- $\phi_3(x, y) = \frac{by}{1-a+aby}$ (z-coordinate is unchanged on iteration)

Appendix: Determining 3-level Partial Fixed Points

BUT... working in $\mathbb{R}^3$ is hard!

Solution: Take the intersection of ϕ_1 with ϕ_3, and the intersection of ϕ_2 with ϕ_3 to reduce to $\mathbb{R}^2$.
We get:

- $\phi_1(y, \phi_3(x,y)) = \frac{1-(1-by)^{n_1}}{1-a(1-by)^{n_1}}$ (x, z coordinates are unchanged on iteration)
- $\phi_2(x, \phi_3(x,y)) = \frac{1-(1-bx)(1-b\phi_3(x,y))^{n_2}}{1-a(1-bx)(1-b\phi_3(x,y))^{n_2}}$ (y, z coordinates are unchanged on iteration)

The intersection of these is where $F(x,y,z) = (x,y,z)$.

Appendix: k-level Concavity Induction

- - ϕ_k is concave and is a function from $[0,1]$ to $[0,1]$.
  - ϕ_{k-1} is concave, non-decreasing in each argument, and is a function from $[0,1]^2$ to $[0,1]$.

  Hence, the composition of ϕ_{k-1} and ϕ_k is concave. Direct inspection shows it is a function from $[0,1]$ to $[0,1]$.

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- - $\phi_{k-1} \circ \phi_k$ is concave and is a function from $[0,1]$ to $[0,1]$.
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- Keep on going for $\phi_{k-3}, \phi_{k-4}, \ldots, \phi_2$.

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  Hence, the composition of ϕ_{k-2} and $\phi_{k-1} \circ \phi_k$ is concave. Direct inspection shows it is a function from $[0,1]$ to $[0,1]$.
- Keep on going for $\phi_{k-3}, \phi_{k-4}, \ldots, \phi_2$.
- Through induction, the composition of $\phi_2, \ldots, \phi_k$ is concave. As ϕ_2 is non-decreasing, so is this composition.

Proofs: $b > (1-a)/\sqrt{n}$

Convergence Case $b > \frac{(1-a)}{\sqrt{n_1+n_2}}$

Theorem (Steven J. Miller, Akihiro Takigawa)

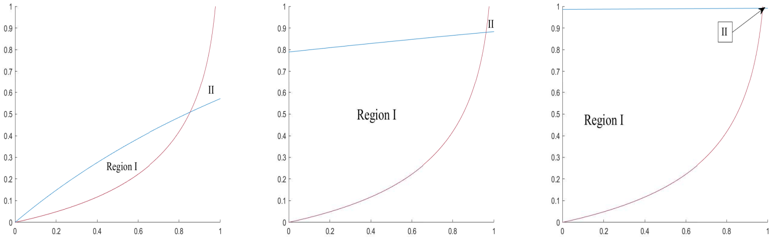
Assume $b > (1-a)/\sqrt{n_1+n_2}$. Then iterates of any point under F converge to the non-trivial fixed point (x_f, y_f, z_f).

Outline of our argument:
Very similar to the $b \leq (1-a)/\sqrt{n_1+n_2}$ case!

- Two regions this time–
  - Region I defined by $x < \phi_1$, $y < \phi_2$ and $z < \phi_3$.
  - Region II defined by $x > \phi_1$, $y > \phi_2$ and $z > \phi_3$.
  - (Note that Region I in the previous case is now Region II.)

  We consider limiting behavior of points in the two regions.
- Then, consider a cuboid with the vertex closest to the origin in Region I, and the vertex furthest from the origin in Region II.
- Finally, use squeeze theorem to show that any point in the cuboid exhibits the same limiting behavior.

$b > \frac{(1-a)}{\sqrt{n_1+n_2}}$: Visualization of Region I and Region II



Red is ϕ_1, blue is ϕ_2.

When $b \leq \frac{1-a}{\sqrt{n_1+n_2}}$, slices of Region I and Region II on the xy-plane at (from left to right) $z=0$, $z=0.25$, $z=0.75$. ($b=0.4$, $n_1=6$, $n_2=10$, $a=0.5$).

Results

Key lemmas (proofs by algebra):

- Points in Region I strictly increase in x, y and z on iteration by F, and points in Region II strictly decrease in x, y and z on iteration.
- Points in Region I iterate inside Region I under F, and points in Region II iterate inside Region II under F.
- All non-trivial points in Regions I and II converge to the non-trivial fixed point under F.

Armed with the above lemmas, we now complete the proof.

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- As the vertices (x_l, y_l, z_l) and (x_u, y_u, z_u) iterate to the non-trivial fixed points (in Regions I and II), so too do all the other points in cuboid.