Volumes of Simplices in Intermediate Dimensions

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Lucas Chen, University of Chicago
Naida Gavranović, University of Cambridge
Joshua Khan, University of Rochester
Connor Yau, Williams College

Mentors: Alex Iosevich, Steven J. Miller, Evyindur Pálsson

Joint with:
Yuval Amit, University of Maryland
Devayani Pradhan, University of Michigan

contacts: joshuakhan1307@gmail.com, lucasch@uchicago.edu, ng558@cam.ac.uk

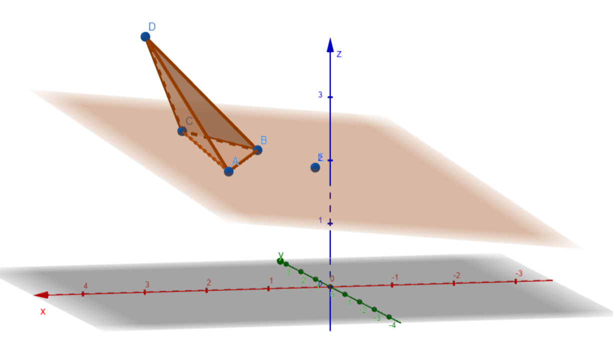
Background

The **Erdös distinct distances** problem asks: given a finite set of points in the plane $\mathbb{R}^2$, what is the number of distinct distances between them?

There are variations of this problem which were explored: Switching to distances in finite fields or noticing that distances are dictated by 2 points, can we look at areas dictated by 3 points?

Background

Definition: We say a set of points $\{x_0, \ldots, x_k\} \subset \mathbb{F}$ is affinely independent if $x_1 - x_0, \ldots, x_k - x_0$ are linearly independent.



The volume of simplices problem

A k-**simplex** in $\mathbb{R}^d$ is the minimal convex set containing $k+1$ affinely independent points. We say that this $k+1$-tuple of points *generates* the k-simplex.

The simplex provides a fairly natural generalization of a line segment, so we study the set of simplex *volumes* generated by a point set.

Recent results

In the continuous Euclidean case:

- Shmerkin and Yavicoli 2025 resolved a sharp bound on a set's fractal dimension to ensure positive Lebesgue measure of the set of volumes of generated simplices of maximal dimension.
- Gaitan and Palsson 2026 extended this result to simplices of dimension between $d/2$ and d in an ambient space $\mathbb{R}^d$.

The finite field setting

Incidence geometry problems often find informative parallels in finite fields. However, there is no defined ”interior” of a simplex in a finite field!

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Iosevich, Rudnev, and Zhai 2015 instead examine a purely algebraic analogue for a point set in $\mathbb{F}_q^2$, defined:

Definition

The **volume** V_d of a d-simplex with vertices $\{x^0, \dots, x^d\} \subset \mathbb{F}_q^d$ is given by:

$$V_d(x^0, \dots, x^d) = \begin{vmatrix} 1 & \dots & 1 \\ x_0^0 & \dots & x_0^d \\ \vdots & \ddots & \vdots \\ x_{d-1}^0 & \dots & x_{d-1}^d \end{vmatrix}.$$

The finite field setting

However, the dimension of the simplex can be mismatched with the ambient dimension. Here the determinant no longer works so we settle on a definition of volume as being Gram determinant.

We aim to improve or generalize the bound of Iosevich, Rudnev, and Zhai 2015. For intermediate dimension we require a slightly different definition via the Gram matrix:

Definition
For $\{x^0, \ldots, x^k\} \subset \mathbb{F}_q^d$ with $k \leq d$, let $u^i = x^i - x^0$.

$$U = \begin{pmatrix} | & & | \\ u^1 & \cdots & u^k \\ | & & | \end{pmatrix}$$

we define the **Gram volume** V_k in terms of the quantity:

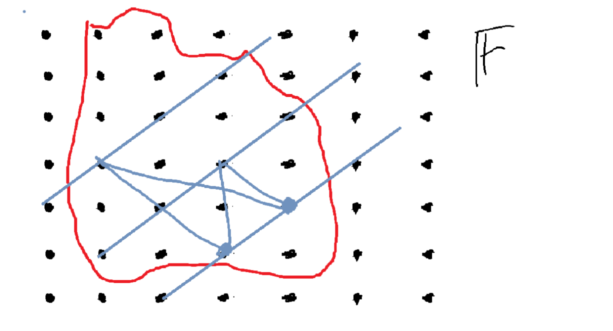
$$\mathsf{V}_k(x^0, \ldots, x^k) = \det U^T U = \det[u^i \cdot u^j]$$

Why this notion of Volume?

But why is this the notion of volume we choose to use? It becomes more natural when we look at the approach in the Ioesevich Rudnev Zhai.

Becks Theorem in the original paper

The initial combinatorial argument of IRZ was talking about how a sufficiently large subset of $\mathbb{F}_q^2$ determines a set of lines with an important property where affine lines determine different volumes



Fourier Analysis and Orthogonal Projections

For the Fourier approach, the set $\nu(t) = |\{(x, y) \in F \times G : x \cdot y = t\}|$, $F, G \subset \mathbb{F}_q^2$ is estimated. An important part of this is the result is taking a subspace and looking at projections onto its orthogonal subspace.

Theorem (Beck's theorem in $\mathbb{F}_q^2$)
Suppose that $E \subset \mathbb{F}_q^2$ with $q \geq C$, for some absolute constant C, and

$$|E| \geq 64q \log q.$$

Then pairs of distinct points of E generate at least $\frac{q^2}{8}$ distinct straight lines in $\mathbb{F}_q^2$. Moreover, at least every fourth point $z \in E$ has the property that it is incident to at least $\frac{q}{4}$ of the latter straight lines, each supporting at least $\frac{|E|}{2q}$ and fewer than $\frac{2|E|}{q}$ points of E, other than z.

Reducing $k+1$ simplices to k simplices

Lemma (Pinned Distance Reduction)
Say that $E \subset \mathbb{F}_q^d$ If we have some $x_0, \ldots, x_k$ affinely independent spanning a non degenerate k flat we call $W = \text{span}\{x_i - x_0\}$. Define $\pi : \mathbb{F}_q^d \to W_\perp$, where $y \in E$ can be decomposed such that $y - x_0 = v_W + v_\perp,\ (v_W, v_\perp) \in W \times W_\perp$ with $\pi(y) = v_\perp$. Then

$$\begin{aligned} |Vol_{k+1}^{(x_0,\ldots,x_k)}(E)| &= |\{Vol_{k+1}(x_0,\ldots,x_k,y) : y \in E\}| \\ &= |\{\pi(y-x_0)\cdot\pi(y-x_0) : y \in E\}|. \end{aligned}$$

Orthogonal projection sizes

We want to find bounds which ensure a large number of heights for some base in the point set. To do so we apply some results on **orthogonal projection**.

For a set $E \subset \mathbb{F}_p^n$ we define the projection $\pi^W(E)$ onto cosets as follows:

$$\pi^W(E) = \{x + W : E \cap (x + W) \neq \emptyset, x \in \mathbb{F}_p^n\}$$

(put image here) We want the size of the projection $\pi^W(E)$ to exceed a fixed number of cosets for enough subspaces W.

Orthogonal projection sizes

Theorem (Chen, 2017)

Let $E \subset \mathbb{F}_p^n$. Then

(a) *for any $N < \frac{1}{2}|E|$,*

$$\left|\left\{W \in G(n, n-m) : |\pi^W(E)| \leq N\right\}\right| \leq 4p^{(n-m)m-m}N;$$

(b) *for any $\delta \in (0,1)$,*

$$\left|\left\{W \in G(n, n-m) : |\pi^W(E)| \leq \delta p^m\right\}\right| \leq 2\left(\frac{\delta}{1-\delta}\right) p^{m(n-m)+m}|E|^{-1}.$$

Beck's theorem in multiple settings

A higher-dimensional analog of Beck's theorem is *not known* for $\mathbb{F}_q$. We consider some potential approaches to proving this analog, similar to some known results.

Theorem (Lund, 2021)

For $E \subset \mathbb{F} := \mathbb{R}$ or $\mathbb{C}$ a set of n points, we have for k less than the "essential dimension" of E:

$$f_k = \Theta\left(\prod_{i=0}^{k}(n - g_i)\right)$$

where f_k is the number of affine k-subspaces generated, and g_i depends on the essential dimension.

Beck's in finite fields

The difficulty of generalizing Beck's in $\mathbb{F}_q$ ends up partly similar to its difficulty in the continuum case. As such, we hope to partially adapt this analogous theorem:

Theorem (Ren, 2022)

Let $X \subset \mathbb{R}^d$ be a Borel set such that $\dim_H(X \setminus H) = \dim_H X$ for all k-planes H. Then, the line set $\mathcal{L}(X)$ spanned by pairs of distinct points in X satisfies

$$\dim_H(\mathcal{L}(X)) \geq \min\{2 \dim_H X, 2k\}.$$

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Thank You

Lucas Chen
University of Chicago
(lucasch@uchicago.edu)

Naida Gavranović
University of Cambridge
(ng558@cam.ac.uk)

Joshua Khan
University of Rochester
(joshuakhan1307@gmail.com)

Connor Yau
Williams College
(cy10@williams.edu)

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