# From M&Ms to Mathematics, or, How I learned to answer questions and help my kids love math.

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Williamstown Elementary School, October 1, 2018

## Some Issues for the Future

- World is rapidly changing – powerful computing cheaply and readily available.
- What skills are we teaching? What skills should we be teaching?
- One of hardest skills: how to think / attack a new problem, how to see connections, what data to gather.

## Plotting Functions: Plot of $f(x) = x + 4$

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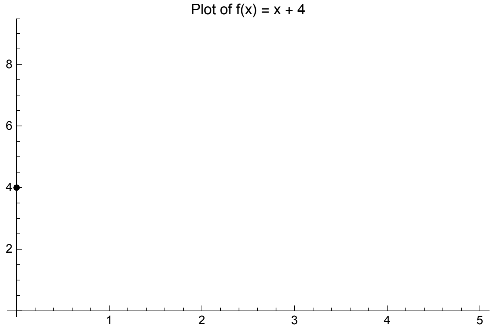


**Figure:** When $x = 1$ we have $f(1) = ?$

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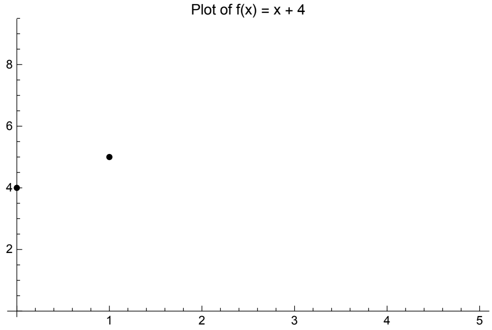


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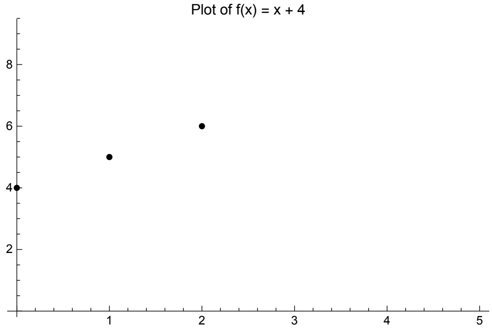


**Figure:** When $x = 3$ we have $f(3) = ?$

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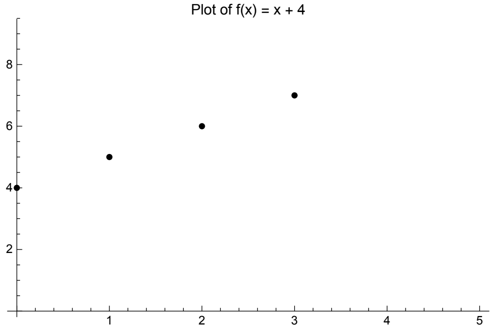


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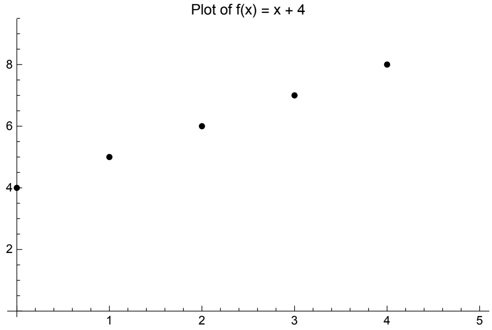


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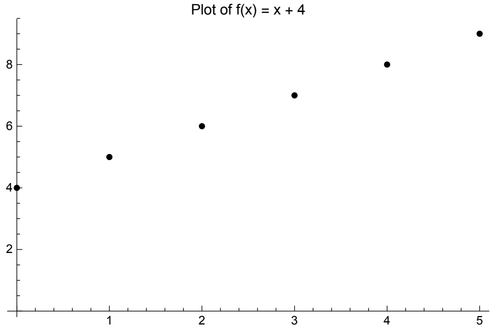


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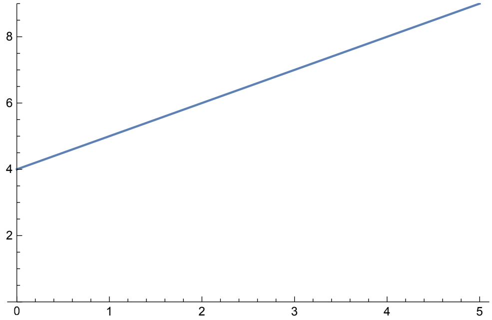

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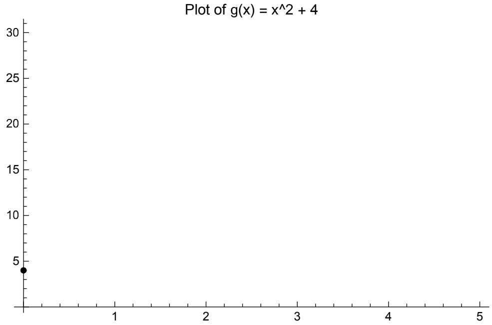


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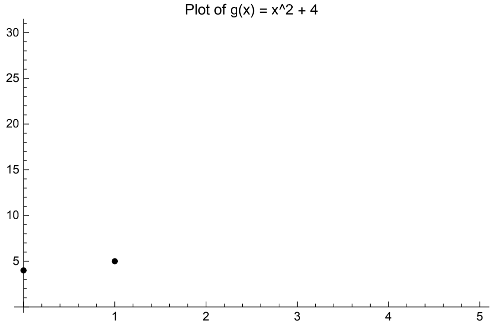


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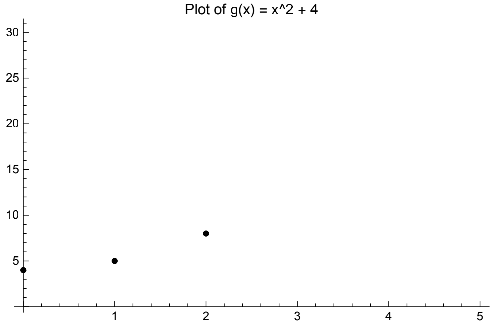


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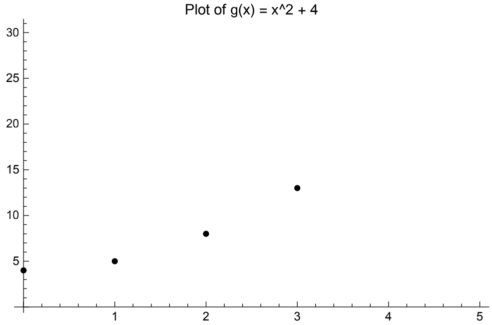


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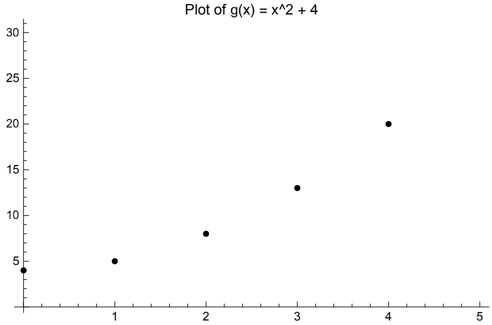


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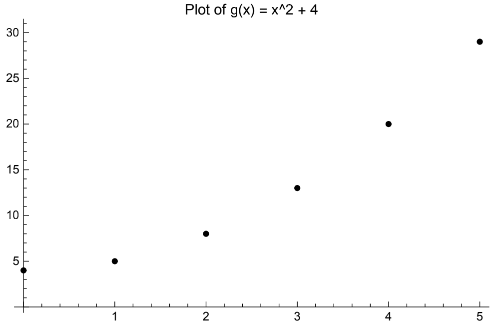


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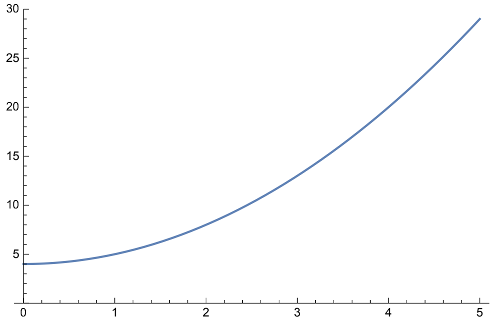

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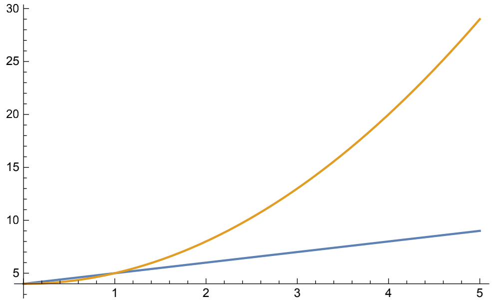
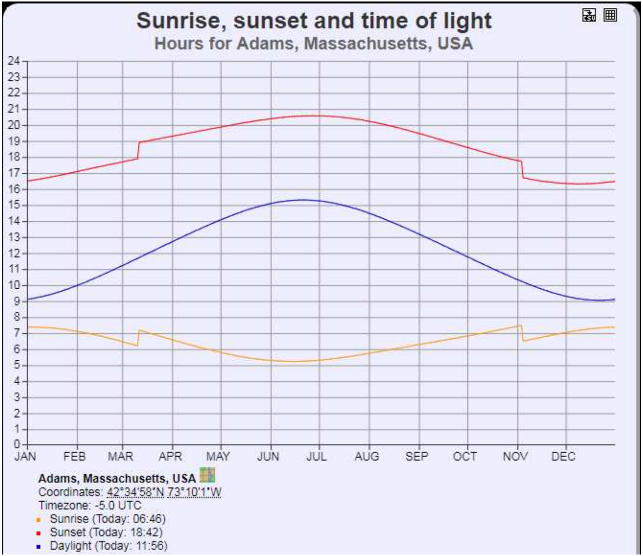


# Functions of the World



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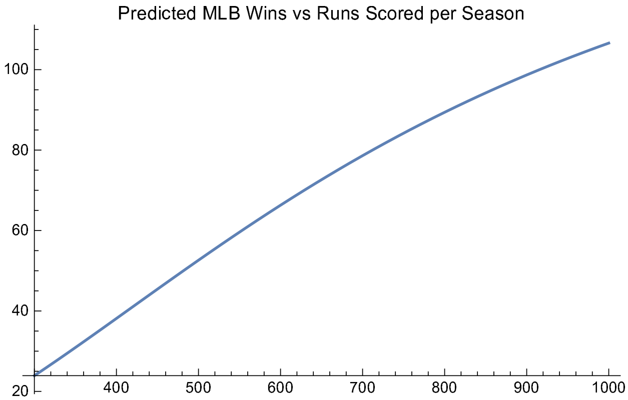

# The M&M Game

## Bacon Numbers

Kevin Bacon game: https://oracleofbacon.org/
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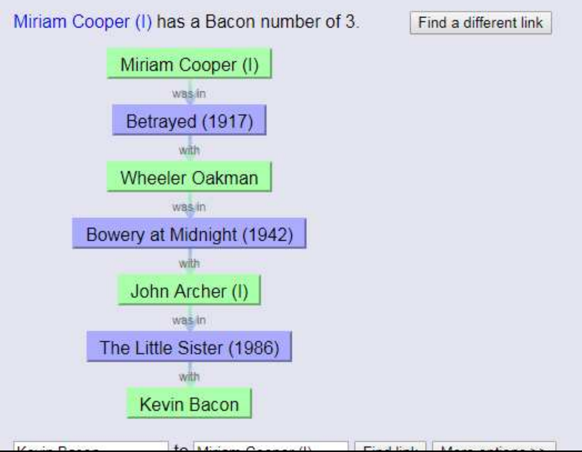
Miriam Cooper from *The Old Shoemaker* (1915, among others):



## Bacon Numbers

Kevin Bacon game: https://oracleofbacon.org/
How are Bacon numbers distributed?

| Kevin Bacon Number | # of People |
|---|---|
| 0 | 1 |
| 1 | 3452 |
| 2 | 403920 |
| 3 | 1504560 |
| 4 | 390201 |
| 5 | 34150 |
| 6 | 4181 |
| 7 | 601 |
| 8 | 119 |
| 9 | 7 |
| 10 | 1 |

# Erdős Numbers

Paul Erdős: 509 co-authors, at least 1416 papers:

| | | | |
|---|---|---|---|
| Erdös number | 0 | --- | 1 person |
| Erdös number | 1 | --- | 504 people |
| Erdös number | 2 | --- | 6593 people |
| Erdös number | 3 | --- | 33605 people |
| Erdös number | 4 | --- | 83642 people |
| Erdös number | 5 | --- | 87760 people |
| Erdös number | 6 | --- | 40014 people |
| Erdös number | 7 | --- | 11591 people |
| Erdös number | 8 | --- | 3146 people |
| Erdös number | 9 | --- | 819 people |
| Erdös number | 10 | --- | 244 people |
| Erdös number | 11 | --- | 68 people |
| Erdös number | 12 | --- | 23 people |
| Erdös number | 13 | --- | 5 people |

Thus the **median Erdös number is 5**; the **mean is 4.65**, and the **standard deviation is 1.21**.

*The M&M Game: From Morsels to Modern Mathematics* (Ivan Badinski, Christopher Huffaker, Nathan McCue, Cameron N. Miller, Kayla S. Miller, Steven J. Miller and Michael Stone), Mathematics Magazine **90** (2017), no. 3, 197–207.

## What Counts as an Erdős Number?

## Erdős-Bacon Numbers

Sum of your Erdős and your Bacon number!

From Wikipedia:

- Mathematician Daniel Kleitman: 3: co-author of Erdős multiple times, Bacon number of 2 from Minnie Driver in Good Will Hunting.
- Danica McKellar (Winnie Cooper in The Wonder Years): 6: math paper gives an Erdős number of 4, Bacon number of 2 from Margaret Easley.
- Natalie Portman (Padmé Amidala): 7.

## Motivating Question

Cam (4 years): `If you're born on the same day, do you die on the same day?`

## M&M Game Rules

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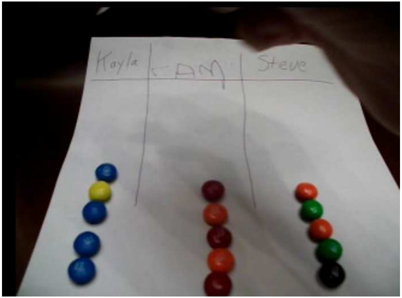


(1) Everyone starts off with $k$ M&Ms (we did 5).

(2) All toss fair coins, eat an M&M if and only if head.

## Be active – ask questions!

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Question 1: How likely is a tie (as a function of $k$)?

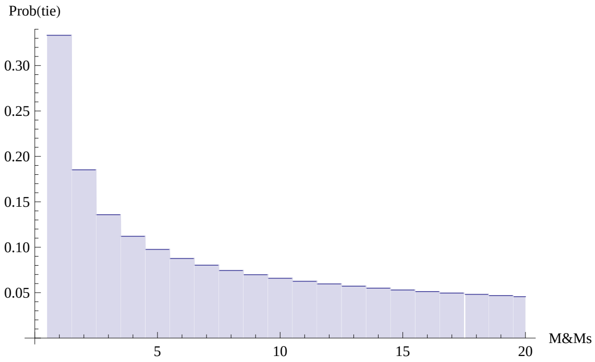
Question 2: How long until one dies?

Question 3: Generalize the game: More people? Biased coin?

Important to ask questions – curiousity is good and to be encouraged! Value to the journey and not knowing the answer.

Let's gather some data! Let's play!

# Probability of a tie in the M&M game (2 players)




Prob(tie) $\approx$ 33% (1 M&M), 19% (2 M&Ms), 14% (3 M&Ms), 10% (4 M&Ms).

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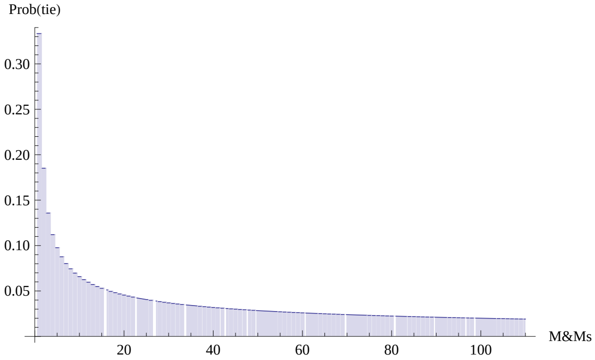


I first gave this talk at a 110th anniversary meeting of the Assoc. of Teachers of Mathematics in Mass....

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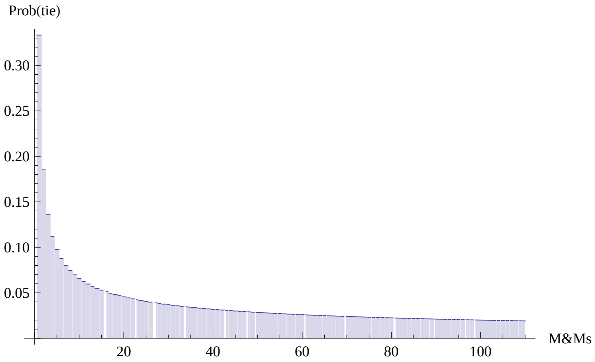


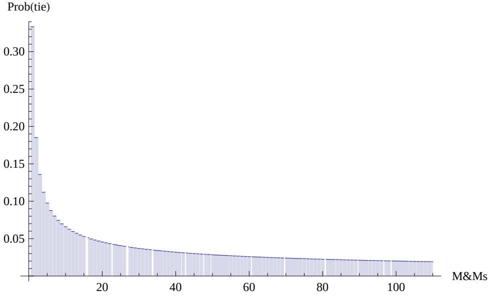

... asked them: what will the next 110 bring us?

Never too early to lay foundations for future classes.

## Welcome to Statistics and Inference!

- **Goal:** Gather data, see pattern, extrapolate.
- **Methods:** Simulation, analysis of special cases.
- **Presentation:** It matters how we show data, and which data we show.

# Viewing M&M Plots




Hard to predict what comes next.

## Introduction to Logarithms

Can write any number as a significand times a power of 10.

- $2018 = 2.018 * 1000 = 2.018 * 10^{3}$.
- $2004 = 2.004 * 1000 = 2.004 * 10^{3}$.
- $.0124 = 1.24 \div 100 = 1.24 * \frac{1}{100} = 1.24 * 10^{-2}$.

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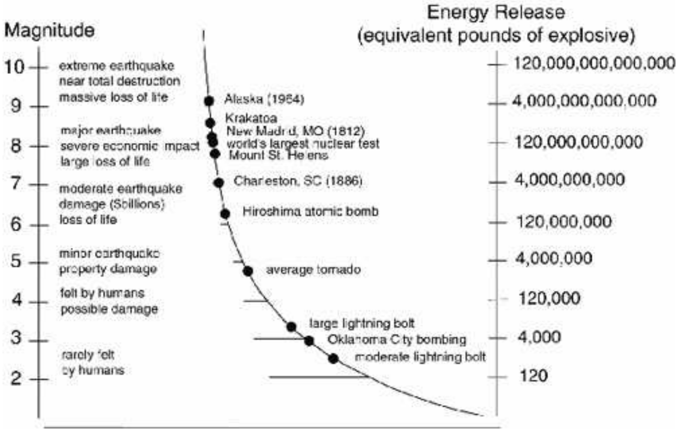
Logarithms have a lot of wonderful properties, including

$$\log_{10}(A * B) \;=\; \log_{10}(A) + \log_{10}(B).$$

If $\log_{10}(A) = \log_{10}(B) + 1$ then $A$ is ten times larger than $B$; if $\log_{10}(A) = \log_{10}(B) + 2$ than $A$ is 100 times larger!

# Richter and Decibel Scales



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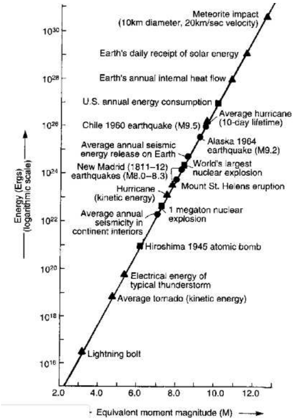

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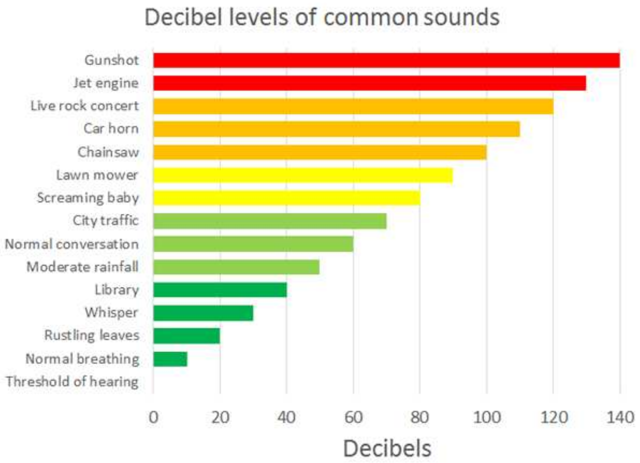
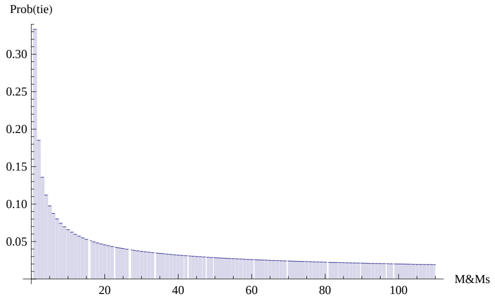
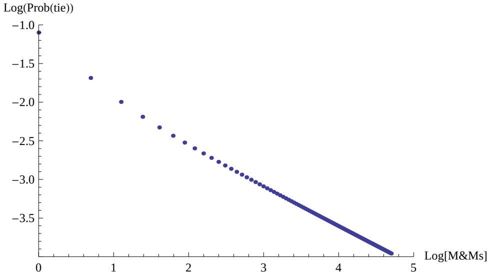


# Viewing M&M Plots



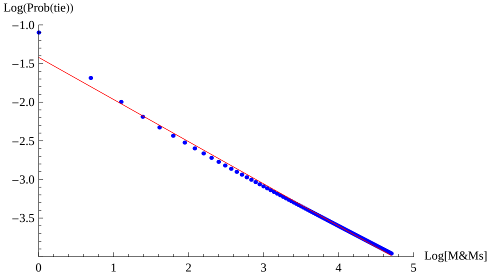

Hard to predict what comes next.

# Viewing M&M Plots: Log-Log Plot




Logarithms are useful! Can see relationships.

## Viewing M&M Plots: Log-Log Plot




Best fit line:

$\log(\text{Prob}(\text{tie})) = -1.42022 - 0.545568 \log(\#\text{M\&Ms})$ or

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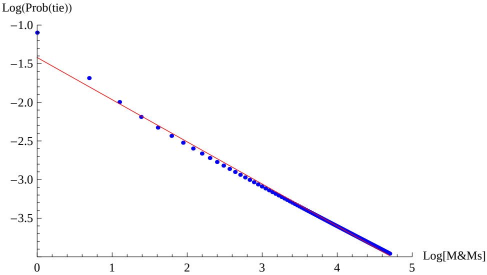


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Predicts probability of a tie when $k = 220$ is 0.01274, but answer is 0.0137. **What gives?**

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Small values can mislead / distort. Let's go from $k = 50$ to 110.

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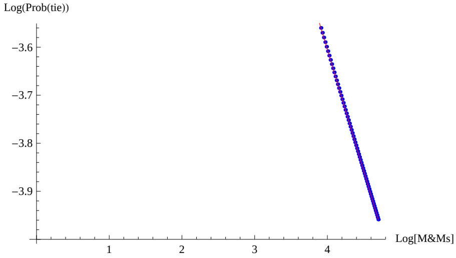


Best fit line:

$\log(\text{Prob}(\text{tie})) \;=\; -1.58261 - 0.50553\log(\#\text{M\&Ms})$ or
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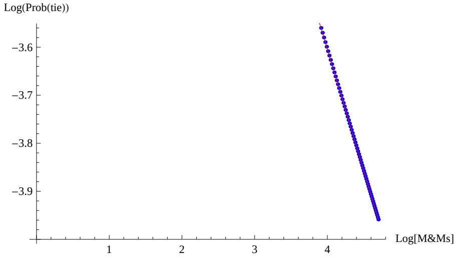


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Get 0.01344 for $k = 220$ (answer 0.01347); **much better!**

From Shooting Hoops
to the Geometric Series Formula

## Simpler Game: Hoops

Game of hoops: first basket wins, alternate shooting.

## Simpler Game: Hoops: Mathematical Formulation

**Bird** and **Magic** (I'm old!) alternate shooting; first basket wins.

- **Bird** always gets basket with probability $p$.
- **Magic** always gets basket with probability $q$.

Let $x$ be the probability **Bird** wins – what is $x$?

## Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

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Break into cases:

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- **Bird** wins on 3rd shot: $(1-p)(1-q) \cdot (1-p)(1-q) \cdot p$.

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- **Bird** wins on 3rd shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdot p$.
- **Bird** wins on nth shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

## Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: $p$.
- **Bird** wins on 2nd shot: $(1-p)(1-q)\cdot p$.
- **Bird** wins on 3rd shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdot p$.
- **Bird** wins on nth shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Let $r=(1-p)(1-q)$. Then

$$\begin{aligned} x &= \text{Prob}(\textbf{Bird}\text{ wins}) \\ &= p+rp+r^2p+r^3p+\cdots \\ &= p\left(1+r+r^2+r^3+\cdots\right), \end{aligned}$$

the geometric series.

## Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird}\text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

## Solving the Hoop Game: The Power of Perspective

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$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

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Have

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$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1-p)(1-q)$$

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Have

$$x = \text{Prob}(\textbf{Bird}\ \text{wins}) = p + (1 - p)(1 - q)x$$

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Showed

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will solve without the geometric series formula.

Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1-p)(1-q)x = p + rx.$$

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Thus

$$(1-r)x = p \quad \text{or} \quad x = \frac{p}{1-r}.$$

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$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

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Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1-p)(1-q)x = p + rx.$$

Thus

$$(1-r)x \;=\; p \quad \text{or} \quad x \;=\; \frac{p}{1-r}.$$

As $x = p(1 + r + r^2 + r^3 + \cdots)$, find

$$1 + r + r^2 + r^3 + \cdots \;=\; \frac{1}{1-r}.$$

## Lessons from Hoop Problem

- Power of Perspective: Memoryless process.
- Can circumvent algebra with deeper understanding! (Hard)
- Depth of a problem not always what expect.
- Importance of knowing more than the minimum: connections.
- Math is fun!

# The M&M Game

## Solving the M&M Game

Overpower with algebra: Assume $k$ M&Ms, two people, fair coins:

$$\text{Prob(tie)} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2},$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is a binomial coefficient.

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is a binomial coefficient.

"Simplifies" to $4^{-k}\ {}_2F_1(k, k, 1, 1/4)$, a special value of a hypergeometric function! (Look up / write report.)

A look at your future classes, but is there a better way?

## Solving the M&M Game (cont)

Where did formula come from? Each turn one of four equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.
- Neither eat.

Probability of each event is 1/4 or 25%.

## Solving the M&M Game (cont)

Where did formula come from? Each turn one of four equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.
- Neither eat.

Probability of each event is 1/4 or 25%.
Each person has exactly $k-1$ heads in first $n-1$ tosses, then ends with a head.

$$\text{Prob(tie)} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2}.$$

## Solving the M&M Game (cont)

Use the lesson from the Hoops Game: Memoryless process!

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If neither eat, as if toss didn't happen. Now game is finite.

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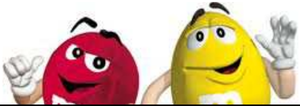
If neither eat, as if toss didn't happen. Now game is finite.

Much better perspective: each "turn" one of three equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.

Probability of each event is 1/3 or about 33%

$$\sum_{n=0}^{k-1} \binom{2k-n-2}{n} \left(\frac{1}{3}\right)^{n} \binom{2k-2n-2}{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \binom{1}{1} \frac{1}{3}.$$



## Solving the M&M Game (cont)

Interpretation: Let Cam have $c$ M&Ms and Kayla have $k$; write as $(c, k)$.

Then each of the following happens 1/3 of the time after a 'turn':

- $(c, k) \longrightarrow (c-1, k-1)$.
- $(c, k) \longrightarrow (c-1, k)$.
- $(c, k) \longrightarrow (c, k-1)$.

## Solving the M&M Game (cont): Assume $k = 4$: First Step

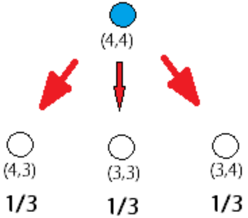



**Figure:** The M&M game when $k = 4$, going down one level.

## Solving the M&M Game (cont): Assume $k = 4$: Full Gory!

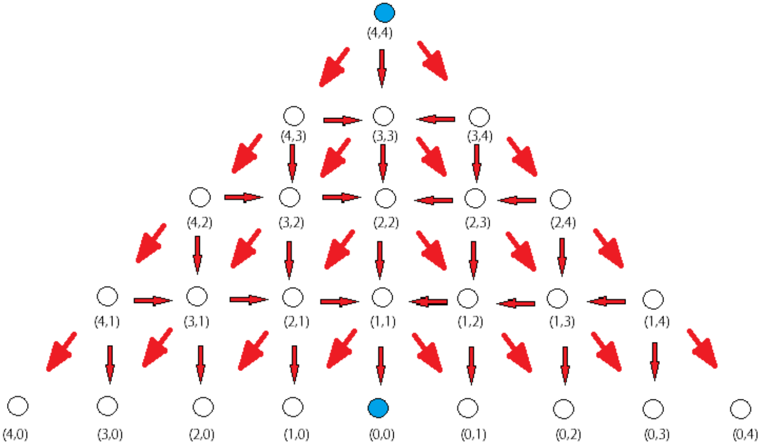



**Figure:** The M&M game when $k = 4$. Count the paths! Answer 1/3 of probability hit (1,1).

# Solving the M&M Game (cont): Assume $k = 4$: Full Gory!

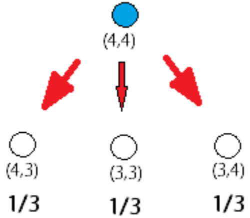



**Figure:** The M&M game when $k = 4$, going down one level.

## Solving the M&M Game (cont): Assume $k = 4$: Full Gory!

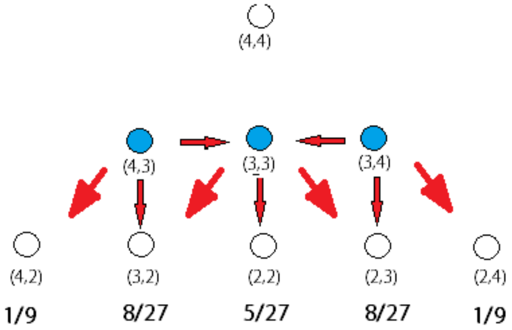



**Figure:** The M&M game when $k = 4$, removing probability from the second level.

## Solving the M&M Game (cont): Assume $k = 4$: Full Gory!

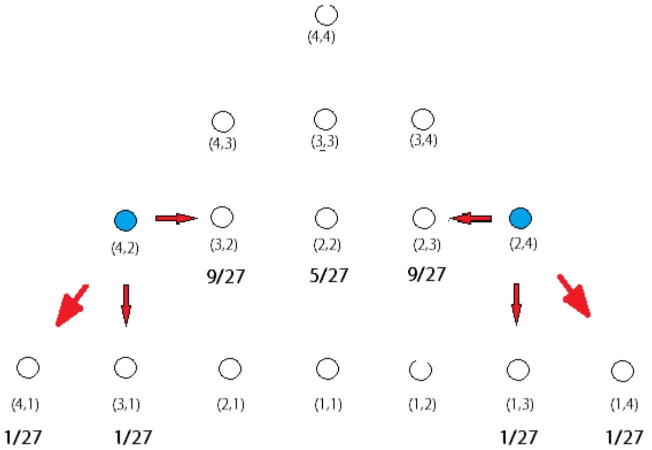



**Figure:** Removing probability from two outer on third level.

## Solving the M&M Game (cont): Assume $k = 4$: Full Gory!

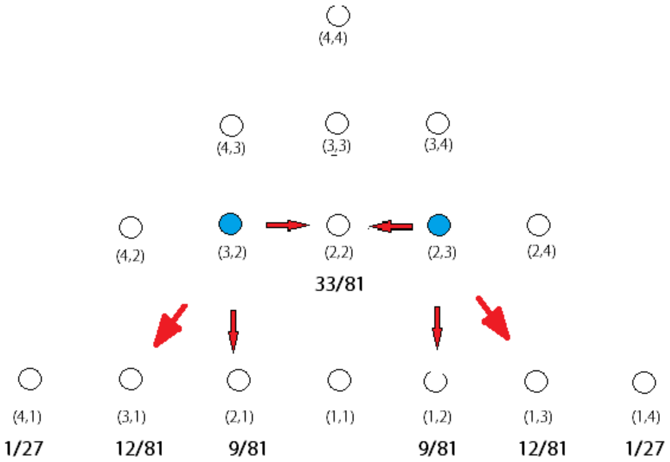



**Figure:** Removing probability from the (3,2) and (2,3) vertices.

# Solving the M&M Game (cont): Assume $k = 4$: Full Gory!

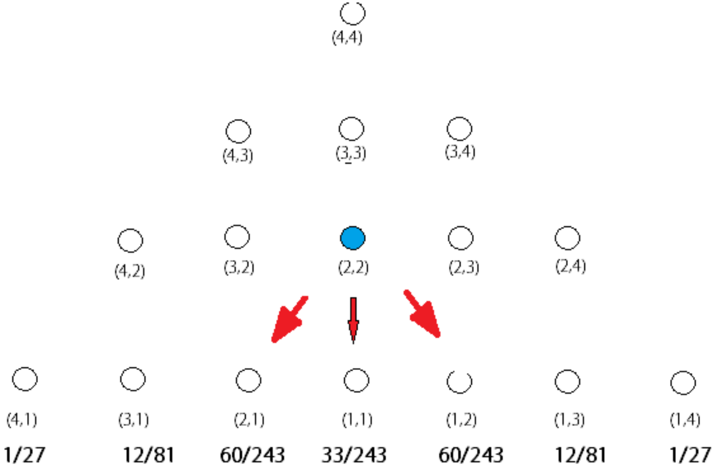



**Figure:** Removing probability from the (2,2) vertex.

## Solving the M&M Game (cont): Assume $k = 4$: Full Gory!

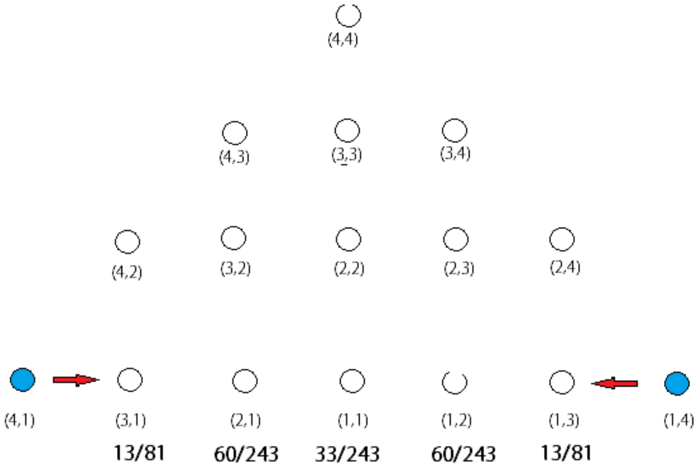



**Figure:** Removing probability from the (4,1) and (1,4) vertices.

# Solving the M&M Game (cont): Assume $k = 4$: Full Gory!

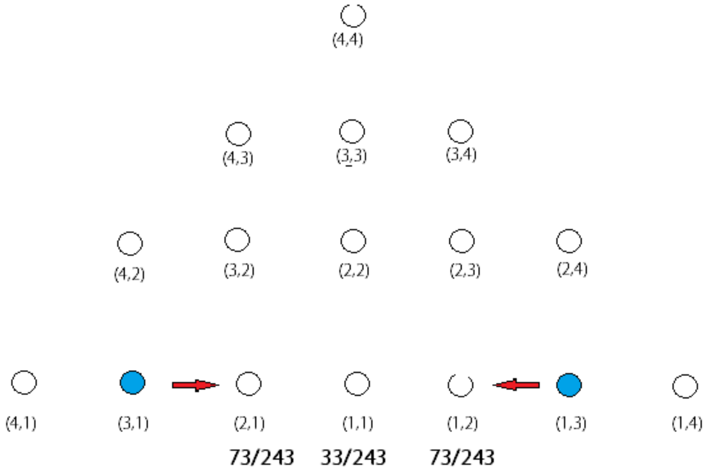



**Figure:** Removing probability from the (3,1) and (1,3) vertices.

## Solving the M&M Game (cont): Assume $k = 4$: Full Gory!

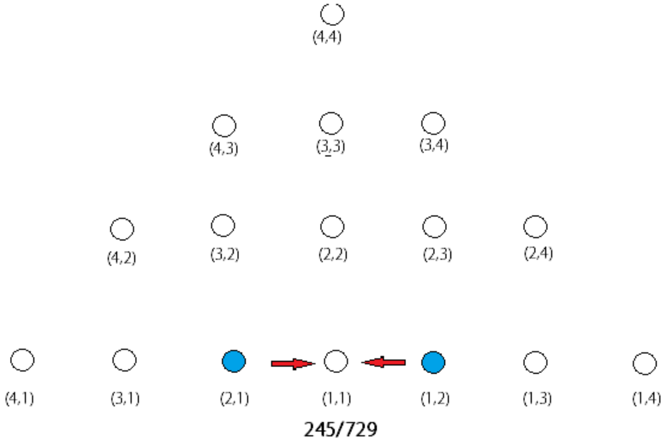



**Figure:** Removing probability from (2,1) and (1,2) vertices. Answer is 1/3 of (1,1) vertex, or 245/2187 (about 11%).

## Interpreting Proof: Connections to the Fibonacci Numbers!

Fibonaccis: $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0, F_1 = 1$.

Starts 0, 1, 1, 2, 3, 5, 8, 13, 21, .... http://www.youtube.com/watch?v=kkGeOWYOFoA.

Binet's Formula (can prove via 'generating functions'):

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

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M&Ms: For $c, k \geq 1$: $x_{c,0} = x_{0,k} = 0$; $x_{0,0} = 1$, and if $c, k \geq 1$:

$$x_{c,k} \; = \; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

Reproduces the tree but a lot 'cleaner'.

## Interpreting Proof: Finding the Recurrence

What if we didn't see the 'simple' recurrence?

$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1}+\frac{1}{3}x_{c-1,k}+\frac{1}{3}x_{c,k-1}.$$

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The following recurrence is 'natural':

$$x_{c,k} \;=\; \frac{1}{4}x_{c,k} + \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1}.$$

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Obtain 'simple' recurrence by algebra: subtract $\frac{1}{4}x_{c,k}$:

$$\begin{aligned} \frac{3}{4}x_{c,k} &= \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1} \\ \text{therefore } x_{c,k} &= \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}. \end{aligned}$$

## Solving the Recurrence

$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

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$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

- $x_{0,0} = 1$.

## Solving the Recurrence

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

- $x_{0,0} = 1$.
- $x_{1,0} = x_{0,1} = 0$.
- $x_{1,1} = \frac{1}{3}x_{0,0} + \frac{1}{3}x_{0,1} + \frac{1}{3}x_{1,0} = \frac{1}{3} \approx 33.3\%$.

## Solving the Recurrence

$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

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- $x_{2,0} = x_{0,2} = 0$.
- $x_{2,1} = \frac{1}{3}x_{1,0} + \frac{1}{3}x_{1,1} + \frac{1}{3}x_{2,0} = \frac{1}{9} \;=\; x_{1,2}$.
- $x_{2,2} = \frac{1}{3}x_{1,1} + \frac{1}{3}x_{1,2} + \frac{1}{3}x_{2,1} = \frac{1}{9} + \frac{1}{27} + \frac{1}{27} \;=\; \frac{5}{27} \approx 18.5\%$.

## Try Simpler Cases!!!

Try and find an easier problem and build intuition.

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Walking from (0,0) to $(k, k)$ with allowable steps (1,0), (0,1) and (1,1), hit $(k, k)$ before hit top or right sides.

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Walking from (0,0) to $(k,k)$ with allowable steps (1,0), (0,1) and (1,1), hit $(k,k)$ before hit top or right sides.

Generalization of the Catalan problem. There don't have (1,1) and stay on or below the main diagonal.

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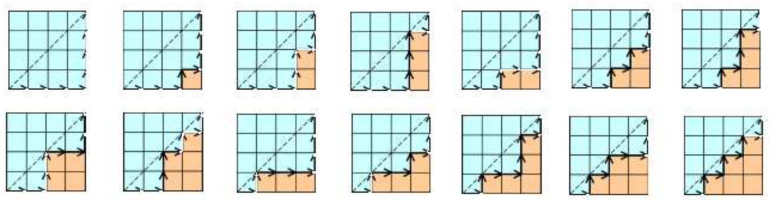

Interpretation: Catalan numbers are valid placings of ( and ).

## Aside: Fun Riddle Related to Catalan Numbers

Young Saul, a budding mathematician and printer, is making himself a fake ID. He needs it to say he's 21. The problem is he's not using a computer, but rather he has some symbols he's bought from the store, and that's it. He has one 1, one 5, one 6, one 7, and an unlimited supply of + - $*$ / (the operations addition, subtraction, multiplication and division). Using each number exactly once (but you can use any number of +, any number of -, ...) how, oh how, can he get 21 from 1,5, 6,7? Note: you can't do things like 15+6 = 21. You have to use the four operations as 'binary' operations: ( (1+5)$*$6 ) + 7. Problem submitted by ohadbp@infolink.net.il, phrasing by yours truly.

Solution involves valid sentences: $((w+x)+y)+z$, $w+((x+y)+z)$, $\dots$.

For more riddles see my riddles page: `http://mathriddles.williams.edu/`.

## Examining Probabilities of a Tie

When $k = 1$, Prob(tie) = 1/3.

When $k = 2$, Prob(tie) = 5/27.

When $k = 3$, Prob(tie) = 11/81.

When $k = 4$, Prob(tie) = 245/2187.

When $k = 5$, Prob(tie) = 1921/19683.

When $k = 6$, Prob(tie) = 575/6561.

When $k = 7$, Prob(tie) = 42635/531441.

When $k = 8$, Prob(tie) = 355975/4782969.

## Examining Ties: Multiply by $3^{2k-1}$ to clear denominators.

When $k = 1$, get 1.

When $k = 2$, get 5.

When $k = 3$, get 33.

When $k = 4$, get 245.

When $k = 5$, get 1921.

When $k = 6$, get 15525.

When $k = 7$, get 127905.

When $k = 8$, get 1067925.

# OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ....

## OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ....

OEIS: http://oeis.org/.

## OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ....

OEIS: http://oeis.org/.

Our sequence: http://oeis.org/A084771.

The web exists! Use it to build conjectures, suggest proofs....

# OEIS (continued)

A084771 Coefficients of 1/sqrt(1-10*x+9*x^2); also, a(n) is the central coefficient of (1+5*x+4*x^2)^n. 5

1, 5, 33, 245, 1921, 15525, 127905, 1067925, 9004545, 76499525, 653808673, 5614995765, 48416454529, 418895174885, 3634723102113, 31616937184725, 275621102802945, 2407331941640325, 21061836725455905, 184550106298084725 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS Also number of paths from (0,0) to (n,0) using steps U=(1,1), H=(1,0) and D=(1,-1), the U steps come in four colors and the H steps come in five colors. - N-E. Fahssi, Mar 30 2008

Number of lattice paths from (0,0) to (n,n) using steps (1,0), (0,1), and three kinds of steps (1,1). [Joerg Arndt, Jul 01 2011]

Sums of squares of coefficients of (1+2*x)^n. [Joerg Arndt, Jul 06 2011]

The Hankel transform of this sequence gives A103488 . - Philippe DELEHAM, Dec 02 2007

REFERENCES Paul Barry and Aoife Hennessy, Generalized Narayana Polynomials, Riordan Arrays, and Lattice Paths, Journal of Integer Sequences, Vol. 15, 2012, #12.4.8.- From N. J. A. Sloane, Oct 08 2012

Michael Z. Spivey and Laura L. Steil, The k-Binomial Transforms and the Hankel Transform, Journal of Integer Sequences, Vol. 9 (2006), Article 06.1.1.

LINKS Table of n, a(n) for n=0..19.

Tony D. Noe, On the Divisibility of Generalized Central Trinomial Coefficients, Journal of Integer Sequences, Vol. 9 (2006), Article 06.2.7.

FORMULA G.f.: 1/sqrt(1-10*x+9*x^2).

Binomial transform of A059304. G.f.: Sum_{k>=0} binomial(2*k, k)*(2*x)^k/(1-x)^(k+1). E.g.f.: exp(5*x)*BesselI(0, 4*x). - Vladeta Jovovic (vladeta(AT)eunet.rs), Aug 20 2003

a(n) = sum(k=0..n, sum(j=0..n-k, C(n,j)*C(n-j,k)*C(2*n-2*j,n-j) ) ). - Paul Barry, May 19 2006

a(n) = sum(k=0..n, 4^k*(C(n,k))^2 ) [From heruneedollar (heruneedollar(AT)gmail.com), Mar 20 2010]

Asymptotic: a(n) ~ 3^(2*n+1)/(2*sqrt(2*Pi*n)). [Vaclav Kotesovec, Sep 11 2012]

Conjecture: n*a(n) +5*(-2*n+1)*a(n-1) +9*(n-1)*a(n-2)=0. - R. J. Mathar,

# Takeaways

## Lessons

- Always ask questions.
- Many ways to solve a problem.
- Experience is useful and a great guide.
- Need to look at the data the right way.
- Often don't know where the math will take you.
- Value of continuing education: more math is better.
- Connections: My favorite quote: `If all you have is a hammer, pretty soon every problem looks like a nail.`