Lower Order Terms for the Variance of Gaussian Primes Across Sectors

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Gaussian Primes

An odd prime $p$ is said to be a **sum of squares** if

$$p = a^2 + b^2.$$ 

*Reciprocity Law (Fermat): An odd prime $p$ is a sum of squares* if and only if $p = 1 \mod 4$.

Examples:
- 3 is not a sum of squares
- $5 = 2^2 + 1^2$ is a sum of squares
- $13 = 2^2 + 3^2$ is a sum of squares
Motivation – Gaussian Primes

Equivalently, an odd prime $p$ is a sum of squares if and only if it **splits** in the Gaussian Integers.

**Gaussian Integers.**

$Z[i] = \{a + bi : a, b \in Z\}$

Examples:
- $3$ does not split
- $5 = (2 + i)(2 - i) = 2^2 + 1^2$ **splits**
- $13 = (2 + 3i)(2 - 3i) = 2^2 + 3^2$ **splits**

3, $2 + i$, $2 + 3i$, etc. are called **Gaussian Primes.**
Gaussian Primes

**Angle of a Gaussian prime:** For an odd prime $p$ such that

$$p = a^2 + b^2 = (a + bi)(a - bi)$$

let $e^{i\theta_p}$ denote the argument of $a + bi$ so that

$$\theta_p = \tan^{-1} \left( \frac{b}{a} \right)$$

By convention we choose $0 \leq b \leq a$ so that $\frac{b}{a} \in [0,1]$, i.e. $\theta_p \in [0, \frac{\pi}{4}]$.

Examples:
- $5 = 2^2 + 1^2 \Rightarrow (a, b) = (2,1) \Rightarrow \theta_5 = \tan^{-1} \left( \frac{1}{2} \right)$
- $13 = 3^2 + 2^2 \Rightarrow (a, b) = (3,2) \Rightarrow \theta_{13} = \tan^{-1} \left( \frac{2}{3} \right)$
Gaussian Primes

Are the Gaussian Primes “randomly” distributed?

Do the first X Gaussian prime angles have the same statistics as X random points in $[0, \frac{\pi}{4})$?

“Random Points” – picked independently and uniformly in $[0, \frac{\pi}{4})$

Angular distribution $(a + ib)/\sqrt{p}$ of the 67 primes $1000 < p < 2000$, $p = 1 \mod 4$
Gaussian Primes

**Uniform Distribution**

Let $\theta_n$ denote the angle associated with the $n^{th}$ prime $p \equiv 1 \mod 4$.

**Hecke (1918):** The angles of Gaussian primes are uniformly distributed: for fixed $0 \leq \alpha \leq \beta \leq \frac{\pi}{4}$,

$$\lim_{N \to \infty} \frac{\# \{n \leq N: \theta_n \in [\alpha, \beta]\}}{N} = \frac{\beta - \alpha}{\pi/4}.$$
Variance of Gaussian Primes

Divide $[0, \pi/4]$ into $K$ small arcs, and ask how many of the $N$ prime angles fall into each arc. Let

$$N_{L,N}(\theta) = \# \left\{ n \leq N : \theta_n \in \left[ \theta, \theta + \frac{\pi/4}{K} \right] \right\}.$$  

The expected value is found to be

$$\langle N_{K,N}(\theta) \rangle = \frac{1}{\pi/4} \int_{0}^{\pi/4} N_{K,N}(\theta) \, d\theta \sim \frac{N}{K}.$$
Variance of Gaussian Primes

The limiting variance is then given by

$$Var(N_{K,N}) = \lim_{N \to \infty} \frac{1}{\pi/4} \int_{0}^{\pi/4} \left| N_{K,N}(\theta) - \frac{N}{K} \right|^2 d\theta.$$ 

In the region $K \gg N^{1+o(1)}$, i.e. very short intervals, we can calculate

$$Var(N_{K,N}) \sim \frac{N}{K}.$$ 

**Theorem (Zeev Rudnick & EW):** Under GRH,

$$Var(N_{K,N}) \ll \frac{N}{K} (\log K)^2.$$ 

For $K \ll N$ the asymptotics remain open.
Conjecture: \( \text{Var}(N_{K,N}) \sim \frac{N}{K} \min(1,2 \frac{\log K}{\log N}) \)

Compare: For \( N \) random points, \( \text{Var}(N_{K,N}^{\text{random}}) \sim \frac{N}{K} \)

Motivations for Conjecture:

a) Numerical Evidence

b) Random Matrix Model: Express variance through zeros of a certain family of Hecke L-functions, then replace these zeros by eigenphases of a suitable ensemble of random matrices.

c) Function Field Analogue
Variance of Gaussian Primes

Question: Can we come up with a conjecture that fits the data even better?

Answer: Yes! Using the Ratios Conjecture

Data: 35241 angles of the Gaussian primes \(10^8 < p < 2 \times 10^8\)

\[
\frac{\text{Var}(N_{K,N}, N)}{N/K} \text{ vs. } \frac{\log K}{\log N}
\]
- Conrey-Farmer-Zinbauer (2007): conjecture an algorithm for computing the averages of products of L-functions over families
Ratios Conjecture – A History


- Applications:
  - n-level correlations and n-level densities
  - Vanishing at the central point
  - Moments of L-functions
Ratios Recipe

- Use approximate functional equation to expand each term in the numerator (ignoring remainder term)

\[ L_k(s) = \sum_{n<x} \frac{a_n(k)}{n^s} + \epsilon_k(s) \sum_{m<y} \frac{a_m(k)}{m^{1-s}} + \text{remainder} \]

- Expand the denominator by the generalized Mobius function

\[ L_k(s) = \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s} \]

- Replace each summand by its expected value when averaged over the family
- Replace each product of \( \epsilon_k(s) \) factors by its expected value when averaged over the family
Hecke Größencharakters

In our case, we apply the conjecture to a family of L-functions attached to Hecke characters (Größencharakters). For a nonzero ideal \( \alpha \in \mathbb{Z}[i] \), define

\[
\chi_k(\alpha) = e^{i4k\theta_{\alpha}}
\]

which is well-defined on ideals. The associated (Hecke) L-function is defined as

\[
L_k(s) = \sum_{\alpha \neq 0} \frac{\chi_k(\alpha)}{N(\alpha)^s} = \prod_{p \text{ prime}} (1 - \chi_k(p)N(p)^{-s})^{-1}
\]

for \( \text{Re}(s) > 1 \), where

\[
N(\alpha) := #(\mathbb{Z}[i]/\alpha)
\]

The Größencharakters can be used to detect elements in a given sector.
Ratios Conjecture

Ratios Conjecture Prediction: let

\[ Y(\alpha, \beta, \gamma, \delta) = \frac{\zeta(1+2\alpha)\zeta(1+\gamma+\delta)\zeta(1+\alpha+\beta)}{\zeta(1+\alpha+\gamma)\zeta(1+\beta+\gamma)\zeta(1+\alpha+\delta)} \times \frac{L(1+2\gamma)L(1+2\delta)L(1+\gamma+\delta)L(1+\alpha+\beta)}{L(1+\alpha+\gamma)L(1+\beta+\gamma)L(1+\beta+\delta)L(1+\alpha+\delta)} \]

Then up to very good approximation,

\[ \sum_{k \in K} \frac{L_k(\frac{1}{2}+\alpha)L_k(\frac{1}{2}+\beta)}{L_k(\frac{1}{2}+\gamma)L_k(\frac{1}{2}+\delta)} = Y(\alpha, \beta, \gamma, \delta) + \frac{1}{1-2\alpha} \left( \frac{\pi}{2K} \right)^{2\alpha} Y(-\alpha, \beta, \gamma, \delta) + \frac{1}{1-2\beta} \left( \frac{\pi}{2K} \right)^{2\beta} Y(\alpha, -\beta, \gamma, \delta) \]

\[ + \frac{1}{1-2\beta} \left( \frac{\pi}{2K} \right)^{2\beta} Y(\alpha, -\beta, \gamma, \delta) + \frac{1}{1-2(\alpha+\beta)} \left( \frac{\pi}{2K} \right)^{2(\alpha+\beta)} Y(-\alpha, -\beta, \gamma, \delta) \]
A smoothed number variance

Define

$$\psi_{K,X}(\theta) = \sum_{\alpha \in \mathbb{Z}[i]} \Phi\left(\frac{N(\alpha)}{X}\right) \Lambda(\alpha) F_K(\theta\alpha - \theta)$$

- $F_K(\theta)$ is a smooth window function, localized at scale $1/K$, $\frac{\pi}{2}$ periodic.
- $\Phi$ is a smooth cut-off for the norm
- $\Lambda(\alpha)$ is the von-Mangoldt function, which is a weighted count of prime powers.

Rather than compute $\text{Var}(N_{K,X})$ directly, we instead study

$$\text{Var}(\psi_{K,X}) = \frac{1}{\pi/2} \int_0^{\frac{\pi}{2}} |\psi_{K,X}(\theta) - \langle \psi_{K,X}(\theta) \rangle|^2 d\theta$$

$F_K, \Phi = \text{fixed window functions}$
A smoothed number variance

By making use of the explicit formula, we may write $V ar(\psi_{K,X})$ in terms of the Hecke L-functions $L_k(s)$:

$$V ar(\psi_{K,X}) = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k \neq 0} e^{4ki\theta} \frac{i}{2\pi k} \hat{f}\left(\frac{k}{K}\right) \int \frac{L_k'}{L_k}(s) \Phi(s) X^s ds \right|^2 d\theta$$

- Opening up the bracket, and inserting the ratios conjecture, gives us a double integral.
A smoothed number variance

The integral then looks as follows:

\[
Var(\psi_{k,x}) = -\frac{K^{\gamma-2}}{4\pi^2} \sum_{k=0} \left| f \left( \frac{k}{K} \right) \right|^2 \left( \int \frac{\partial}{\partial \beta} \frac{\partial}{\partial \alpha} Y(\alpha, \beta, \gamma, \delta) \right)_{y=\alpha, \delta=\beta} \Phi \left( \frac{1}{2} + \alpha \right) \Phi \left( \frac{1}{2} + \beta \right) X^{\alpha} X^{\beta} d\alpha d\beta
\]

\[
+ \left( \int \frac{\partial}{\partial \beta} \frac{\partial}{\partial \alpha} \left( \frac{\pi}{2K} \right)^{2\beta} \frac{1}{1-2\beta} Y(\alpha, -\beta, \gamma, \delta) \right)_{y=\alpha, \delta=\beta} \Phi \left( \frac{1}{2} + \alpha \right) \Phi \left( \frac{1}{2} + \beta \right) X^{\alpha} X^{\beta} d\alpha d\beta
\]

\[
+ \left( \int \frac{\partial}{\partial \beta} \frac{\partial}{\partial \alpha} \left( \frac{\pi}{2K} \right)^{2\alpha} \frac{1}{1-2\alpha} Y(-\alpha, \beta, \gamma, \delta) \right)_{y=\alpha, \delta=\beta} \Phi \left( \frac{1}{2} + \alpha \right) \Phi \left( \frac{1}{2} + \beta \right) X^{\alpha} X^{\beta} d\alpha d\beta
\]

where

\[
Y(\alpha, \beta, \gamma, \delta) = \frac{\zeta(1 + 2\alpha)\zeta(1 + 2\beta)\zeta(1 + \gamma + \delta)\zeta(1 + \alpha + \beta)}{\zeta(1 + \alpha + \gamma)\zeta(1 + \beta + \gamma)\zeta(1 + \alpha + \delta)\zeta(1 + \beta + \delta)} \frac{L(1 + 2\gamma)L(1 + 2\delta)L(1 + \gamma + \delta)L(1 + \alpha + \beta)}{L(1 + \alpha + \gamma)L(1 + \beta + \gamma)L(1 + \alpha + \delta)L(1 + \beta + \delta)L(1 + \alpha + \delta)}
\]
A smoothed number variance

The integral then looks as follows:

\[ \text{Var}(\psi_{\kappa,X}) = - \frac{K^{\gamma - 2}}{4\pi^2} \sum_{k \neq 0} \left| f \left( \frac{k}{K} \right) \right|^2 \left( \int \frac{\partial}{\partial \beta} \frac{\partial}{\partial \alpha} Y(\alpha, \beta, \gamma, \delta) \bigg|_{\gamma = \alpha, \delta = \beta} \Phi \left( \frac{1}{2} + \alpha \right) \Phi \left( \frac{1}{2} + \beta \right) X^\alpha X^\beta d\alpha d\beta \right. \\
+ \left( \int \frac{\partial}{\partial \beta} \frac{\partial}{\partial \alpha} \left( \frac{\pi}{2K} \right)^2 \frac{1}{1 - 2\beta} Y(\alpha, -\beta, \gamma, \delta) \bigg|_{\gamma = \alpha, \delta = \beta} \Phi \left( \frac{1}{2} + \alpha \right) \Phi \left( \frac{1}{2} + \beta \right) X^\alpha X^\beta d\alpha d\beta \right. \\
+ \left( \int \frac{\partial}{\partial \beta} \frac{\partial}{\partial \alpha} \left( \frac{\pi}{2K} \right)^2 \frac{1}{1 - 2\alpha} Y(-\alpha, \beta, \gamma, \delta) \bigg|_{\gamma = \alpha, \delta = \beta} \Phi \left( \frac{1}{2} + \alpha \right) \Phi \left( \frac{1}{2} + \beta \right) X^\alpha X^\beta d\alpha d\beta \right. \]

- The contour has single poles and double poles, coming from the singularity at \( \zeta(1) \) within the ratios conjecture.
- The double poles contribute the main term; the single poles contribute the lower order terms.
New Conjecture

Let $X = K^\gamma$. Then

$$\text{Var}(\psi_{K,X}) = \begin{cases} 
\frac{X}{K} \left( 2 \cdot \log K - \log \frac{\pi^2}{4} \right) & \text{if } \gamma > 2 \\
\frac{X}{K} (\log X - 3) & \text{if } 1 < \gamma < 2 \\
\frac{X}{K} (\log X + 1) & \text{if } 1 < \gamma
\end{cases}$$

Data: 78594 Gaussian primes and prime powers with norm $< 10^6$

$$\frac{\text{Var}(\psi_{K,X})}{X/K} \text{ vs. } 1/\gamma$$
Computing the Variance

Function Fields (Theorem – EW & Rudnick):

\[
\frac{\text{var}(\psi_{K,V})}{X/K} \sim \begin{cases} 
2 \cdot \log_q K - 2, & \gamma > 2 \\
\log_q X - 1 + \eta(\log_q X), & 1 < \gamma < 2 \\
\log_q X + \eta(\log_q X), & \gamma < 1
\end{cases}
\]

Number Fields (Ratios Conjecture):

\[
\frac{\text{var}(\psi_{K,X})}{X/K} \sim \begin{cases} 
2 \cdot \log K - \log \frac{\pi^2}{4}, & \gamma > 2 \\
\log X - 3, & 1 < \gamma < 2 \\
\log X + 1, & \gamma < 1
\end{cases}
\]
Thank you!