Finite conductor models for zeros near the central point of elliptic curve L-functions

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Special Session on Number Theory
AMS Sectional Meeting, Wesleyan University
Middletown, CT, October 2008
Much of this is joint and current work with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith.

Computer programs written with Adam O’Brien, Jon Hsu, Leo Goldmahker, Stephen Lu and Mike Rubinstein, Adam O’Brien.
Outline

- Introduce relevant RMT ensembles.
- Describe results for large conductors.
- Discuss data for small conductors.
- Try to reconcile theory and data.
Random Matrix Ensembles
Orthogonal Random Matrix Models

**RMT:** $SO(2N)$: $2N$ eigenvalues in pairs $e^{\pm i\theta_j}$, probability measure on $[0, \pi]^N$:

$$d\epsilon_0(\theta) \propto \prod_{j<k} (\cos \theta_k - \cos \theta_j)^2 \prod_j d\theta_j.$$
Orthogonal Random Matrix Models

**RMT:** $SO(2N)$: $2N$ eigenvalues in pairs $e^{\pm i\theta_j}$, probability measure on $[0, \pi]^N$:

$$d\epsilon_0(\theta) \propto \prod_{j<k} (\cos \theta_k - \cos \theta_j)^2 \prod_j d\theta_j.$$  

**Independent Model:**

$$\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} I_{2r \times 2r} & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$
RMT: $SO(2N)$: $2N$ eigenvalues in pairs $e^{\pm i \theta_j}$, probability measure on $[0, \pi]^N$:

$$d \epsilon_0(\theta) \propto \prod_{j<k}(\cos \theta_k - \cos \theta_j)^2 \prod_j d\theta_j.$$ 

Independent Model:

$$\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} I_{2r \times 2r} & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$ 

Interaction Model: Sub-ensemble of $SO(2N)$ with the last $2r$ of the $2N$ eigenvalues equal +1: $1 \leq j, k \leq N-r$:

$$d \epsilon_{2r}(\theta) \propto \prod_{j<k}(\cos \theta_k - \cos \theta_j)^2 \prod_j (1 - \cos \theta_j)^{2r} \prod_j d\theta_j,$$
1-Level Density

$L$-function $L(s, f)$: by RH non-trivial zeros $\frac{1}{2} + i\gamma_{f,j}$.

$C_f$: analytic conductor.

$\varphi(x)$: compactly supported even Schwartz fn.

$$D_{1,f}(\varphi) = \sum_j \varphi \left( \frac{\log C_f}{2\pi} \gamma_{f,j} \right)$$
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- individual zeros contribute in limit
- most of contribution is from low zeros
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Katz-Sarnak Conjecture:

$$D_{1,\mathcal{F}}(\varphi) = \lim_{N \to \infty} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{1,f}(\varphi) = \int \varphi(x) \rho_G(\mathcal{F})(x) dx$$

$$= \int \hat{\varphi}(u) \hat{\rho}_G(\mathcal{F})(u) du.$$
Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2}\eta(u).$$

Fourier transform of 1-level density (Rank 2, Indep):

$$\hat{\rho}_{2,\text{Independent}}(u) = \left[ \delta(u) + \frac{1}{2}\eta(u) + 2 \right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2,\text{Interaction}}(u) = \left[ \delta(u) + \frac{1}{2}\eta(u) + 2 \right] + 2(|u| - 1)\eta(u).$$
Comparing the RMT Models

**Theorem: M– ’04**

For small support, one-param family of rank $r$ over $\mathbb{Q}(T)$:

$$\lim_{N \to \infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_j \varphi \left( \frac{\log C_{E_t}}{2\pi} \gamma_{E_t,j} \right) = \int \varphi(x) \rho_{\mathcal{G}}(x) dx + r\varphi(0)$$

where

$$\mathcal{G} = \begin{cases} 
SO & \text{if half odd} \\
SO(\text{even}) & \text{if all even} \\
SO(\text{odd}) & \text{if all odd}
\end{cases}$$

Confirm Katz-Sarnak, B-SD predictions for small support.

Supports Independent and not Interaction model in the limit.
Questions
Interesting Families

Let $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$ be a one-parameter family of elliptic curves of rank $r$ over $\mathbb{Q}(T)$. Natural sub-families:

- Curves of rank $r$.
- Curves of rank $r + 2$. 
Let \( \mathcal{E} : y^2 = x^3 + A(T)x + B(T) \) be a one-parameter family of elliptic curves of rank \( r \) over \( \mathbb{Q}(T) \).

Natural sub-families:

- Curves of rank \( r \).
- Curves of rank \( r + 2 \).

**Question:** Does the sub-family of rank \( r + 2 \) curves in a rank \( r \) family behave like the sub-family of rank \( r + 2 \) curves in a rank \( r + 2 \) family?

Equivalently, does it matter how one conditions on a curve being rank \( r + 2 \)?
Testing Random Matrix Theory Predictions

Know the right model for large conductors, searching for the correct model for finite conductors.

In the limit must recover the independent model, and want to explain data on:

1. **Excess Rank:** Rank $r$ one-parameter family over $\mathbb{Q}(T)$: observed percentages with rank $\geq r + 2$.

2. **First (Normalized) Zero above Central Point:** Influence of zeros at the central point on the distribution of zeros near the central point.
Excess Rank

One-parameter family, rank $r$ over $\mathbb{Q}(T)$.  
Density Conjecture (Generic Family) $\implies$ 50% rank $r$, $r+1$. 
Excess Rank

One-parameter family, rank \( r \) over \( \mathbb{Q}(T) \).
Density Conjecture (Generic Family) \( \implies 50\% \) rank \( r, r+1 \).

For many families, observe
Percent with rank \( r \) \( \approx \) 32%
Percent with rank \( r+1 \) \( \approx \) 48%
Percent with rank \( r+2 \) \( \approx \) 18%
Percent with rank \( r+3 \) \( \approx \) 2%

Problem: small data sets, sub-families, convergence rate log(conductor).
Data on Excess Rank

\[ y^2 + y = x^3 + Tx \]

Each data set 2000 curves from start. Last has conductors of size \(10^{17}\), but on logarithmic scale still small.

<table>
<thead>
<tr>
<th>( t )-Start</th>
<th>Rk 0</th>
<th>Rk 1</th>
<th>Rk 2</th>
<th>Rk 3</th>
<th>Time (hrs)</th>
</tr>
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<tbody>
<tr>
<td>-1000</td>
<td>39.4</td>
<td>47.8</td>
<td>12.3</td>
<td>0.6</td>
<td>&lt;1</td>
</tr>
<tr>
<td>1000</td>
<td>38.4</td>
<td>47.3</td>
<td>13.6</td>
<td>0.6</td>
<td>&lt;1</td>
</tr>
<tr>
<td>4000</td>
<td>37.4</td>
<td>47.8</td>
<td>13.7</td>
<td>1.1</td>
<td>1</td>
</tr>
<tr>
<td>8000</td>
<td>37.3</td>
<td>48.8</td>
<td>12.9</td>
<td>1.0</td>
<td>2.5</td>
</tr>
<tr>
<td>24000</td>
<td>35.1</td>
<td>50.1</td>
<td>13.9</td>
<td>0.8</td>
<td>6.8</td>
</tr>
<tr>
<td>50000</td>
<td>36.7</td>
<td>48.3</td>
<td>13.8</td>
<td>1.2</td>
<td>51.8</td>
</tr>
</tbody>
</table>
Theoretical Results
RMT: Theoretical Results \((N \to \infty, \text{Mean} \to 0.321)\)

Figure 1a: 1st norm. evalue above 1: 23,040 SO(4) matrices
Mean = .709, Std Dev of the Mean = .601, Median = .709
RMT: Theoretical Results \( (N \to \infty, \text{Mean} \to 0.321) \)

Figure 1b: 1st norm. evalune above 1: 23,040 SO(6) matrices
Mean = .635, Std Dev of the Mean = .574, Median = .635
RMT: Theoretical Results ($N \to \infty$)

Figure 1c: 1st norm. evalue above 1: SO(even)
RMT: Theoretical Results ($N \to \infty$)

Figure 1d: 1st norm. evals above 1: SO(odd)
Data
Rank 0 Curves: 1st Normalized Zero above Central Point

Figure 2a: 750 rank 0 curves from
\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \]
\[ \log(\text{cond}) \in [3.2, 12.6], \text{median} = 1.00 \text{ mean} = 1.04, \sigma_\mu = .32 \]
Figure 2b: 750 rank 0 curves from

\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \]

\( \log(\text{cond}) \in [12.6, 14.9] \), median = .85, mean = .88, \( \sigma_\mu = .27 \)
Rank 2 Curves: 1st Norm. Zero above the Central Point

Figure 3a: 665 rank 2 curves from
\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \]
\[ \log(\text{cond}) \in [10, 10.3125], \text{ median } = 2.29, \text{ mean } = 2.30 \]
Rank 2 Curves: 1st Norm. Zero above the Central Point

Figure 3b: 665 rank 2 curves from

\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \]

\[ \log(\text{cond}) \in [16, 16.5], \text{ median } = 1.81, \text{ mean } = 1.82 \]
Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

Figure 4a: 209 rank 0 curves from 14 rank 0 families, 
$\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$
Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median = .81, mean = .86.
### Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

<table>
<thead>
<tr>
<th>Family</th>
<th>Median $\tilde{\mu}$</th>
<th>Mean $\mu$</th>
<th>StDev $\sigma_\mu$</th>
<th>log(cond)</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: $[0,1,1,1,T]$</td>
<td>1.28</td>
<td>1.33</td>
<td>0.26</td>
<td>[3.93, 9.66]</td>
<td>7</td>
</tr>
<tr>
<td>2: $[1,0,0,1,T]$</td>
<td>1.39</td>
<td>1.40</td>
<td>0.29</td>
<td>[4.66, 9.94]</td>
<td>11</td>
</tr>
<tr>
<td>3: $[1,0,0,2,T]$</td>
<td>1.40</td>
<td>1.41</td>
<td>0.33</td>
<td>[5.37, 9.97]</td>
<td>11</td>
</tr>
<tr>
<td>4: $[1,0,0,-1,T]$</td>
<td>1.50</td>
<td>1.42</td>
<td>0.37</td>
<td>[4.70, 9.98]</td>
<td>20</td>
</tr>
<tr>
<td>5: $[1,0,0,-2,T]$</td>
<td>1.40</td>
<td>1.48</td>
<td>0.32</td>
<td>[4.95, 9.85]</td>
<td>11</td>
</tr>
<tr>
<td>6: $[1,0,0,T,0]$</td>
<td>1.35</td>
<td>1.37</td>
<td>0.30</td>
<td>[4.74, 9.97]</td>
<td>44</td>
</tr>
<tr>
<td>7: $[1,0,1,-2,T]$</td>
<td>1.25</td>
<td>1.34</td>
<td>0.42</td>
<td>[4.04, 9.46]</td>
<td>10</td>
</tr>
<tr>
<td>8: $[1,0,2,1,T]$</td>
<td>1.40</td>
<td>1.41</td>
<td>0.33</td>
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<td>9: $[1,0,-1,1,T]$</td>
<td>1.39</td>
<td>1.32</td>
<td>0.25</td>
<td>[7.45, 9.96]</td>
<td>9</td>
</tr>
<tr>
<td>10: $[1,0,-2,1,T]$</td>
<td>1.34</td>
<td>1.34</td>
<td>0.42</td>
<td>[3.26, 9.56]</td>
<td>9</td>
</tr>
<tr>
<td>11: $[1,1,-2,1,T]$</td>
<td>1.21</td>
<td>1.19</td>
<td>0.41</td>
<td>[5.73, 9.92]</td>
<td>6</td>
</tr>
<tr>
<td>12: $[1,1,-3,1,T]$</td>
<td>1.32</td>
<td>1.32</td>
<td>0.32</td>
<td>[5.04, 9.98]</td>
<td>11</td>
</tr>
<tr>
<td>13: $[1,-2,0,T,0]$</td>
<td>1.31</td>
<td>1.29</td>
<td>0.37</td>
<td>[4.73, 9.91]</td>
<td>39</td>
</tr>
<tr>
<td>14: $[-1,1,-3,1,T]$</td>
<td>1.45</td>
<td>1.45</td>
<td>0.31</td>
<td>[5.76, 9.92]</td>
<td>10</td>
</tr>
<tr>
<td><strong>All Curves</strong></td>
<td>1.35</td>
<td>1.36</td>
<td>0.33</td>
<td>[3.26, 9.98]</td>
<td>209</td>
</tr>
<tr>
<td><strong>Distinct Curves</strong></td>
<td>1.35</td>
<td>1.36</td>
<td>0.33</td>
<td>[3.26, 9.98]</td>
<td>196</td>
</tr>
</tbody>
</table>
## Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

<table>
<thead>
<tr>
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<th>StDev $\sigma_{\mu}$</th>
<th>log(cond)</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: [0,1,1,1,T]</td>
<td>0.80</td>
<td>0.86</td>
<td>0.23</td>
<td>[15.02, 15.97]</td>
<td>49</td>
</tr>
<tr>
<td>2: [1,0,0,1,T]</td>
<td>0.91</td>
<td>0.93</td>
<td>0.29</td>
<td>[15.00, 15.99]</td>
<td>58</td>
</tr>
<tr>
<td>3: [1,0,0,2,T]</td>
<td>0.90</td>
<td>0.94</td>
<td>0.30</td>
<td>[15.00, 16.00]</td>
<td>55</td>
</tr>
<tr>
<td>4: [1,0,0,-1,T]</td>
<td>0.80</td>
<td>0.90</td>
<td>0.29</td>
<td>[15.02, 16.00]</td>
<td>59</td>
</tr>
<tr>
<td>5: [1,0,0,-2,T]</td>
<td>0.75</td>
<td>0.77</td>
<td>0.25</td>
<td>[15.04, 15.98]</td>
<td>53</td>
</tr>
<tr>
<td>6: [1,0,0,T,0]</td>
<td>0.75</td>
<td>0.82</td>
<td>0.27</td>
<td>[15.00, 16.00]</td>
<td>130</td>
</tr>
<tr>
<td>7: [1,0,1,-2,T]</td>
<td>0.84</td>
<td>0.84</td>
<td>0.25</td>
<td>[15.04, 15.99]</td>
<td>63</td>
</tr>
<tr>
<td>8: [1,0,2,1,T]</td>
<td>0.90</td>
<td>0.94</td>
<td>0.30</td>
<td>[15.00, 16.00]</td>
<td>55</td>
</tr>
<tr>
<td>9: [1,0,-1,1,T]</td>
<td>0.86</td>
<td>0.89</td>
<td>0.27</td>
<td>[15.02, 15.98]</td>
<td>57</td>
</tr>
<tr>
<td>10: [1,0,-2,1,T]</td>
<td>0.86</td>
<td>0.91</td>
<td>0.30</td>
<td>[15.03, 15.97]</td>
<td>59</td>
</tr>
<tr>
<td>11: [1,1,-2,1,T]</td>
<td>0.73</td>
<td>0.79</td>
<td>0.27</td>
<td>[15.00, 16.00]</td>
<td>124</td>
</tr>
<tr>
<td>12: [1,1,-3,1,T]</td>
<td>0.98</td>
<td>0.99</td>
<td>0.36</td>
<td>[15.01, 16.00]</td>
<td>66</td>
</tr>
<tr>
<td>13: [1,-2,0,T,0]</td>
<td>0.72</td>
<td>0.76</td>
<td>0.27</td>
<td>[15.00, 16.00]</td>
<td>120</td>
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<tr>
<td>14: [-1,1,-3,1,T]</td>
<td>0.90</td>
<td>0.91</td>
<td>0.24</td>
<td>[15.00, 15.99]</td>
<td>48</td>
</tr>
</tbody>
</table>

### All Curves
- Median $\tilde{\mu}$: 0.81
- Mean $\mu$: 0.86
- StDev $\sigma_{\mu}$: 0.29
- log(cond): [15.00, 16.00]
- Number: 996

### Distinct Curves
- Median $\tilde{\mu}$: 0.81
- Mean $\mu$: 0.86
- StDev $\sigma_{\mu}$: 0.28
- log(cond): [15.00, 16.00]
- Number: 863
first set \(\log(\text{cond}) \in [15, 15.5]\); second set \(\log(\text{cond}) \in [15.5, 16]\).

Median \(\hat{\mu}\), Mean \(\mu\), Std Dev (of Mean) \(\sigma_{\mu}\).

<table>
<thead>
<tr>
<th>Family</th>
<th>(\hat{\mu})</th>
<th>(\mu)</th>
<th>(\sigma_{\mu})</th>
<th>Number</th>
<th>(\hat{\mu})</th>
<th>(\mu)</th>
<th>(\sigma_{\mu})</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: ([0,1,3,1,T])</td>
<td>1.59</td>
<td>1.83</td>
<td>0.49</td>
<td>8</td>
<td>1.71</td>
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<td>2: ([1,0,0,1,T])</td>
<td>1.84</td>
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<td>1.81</td>
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<td>3: ([1,0,0,2,T])</td>
<td>2.05</td>
<td>2.03</td>
<td>0.26</td>
<td>16</td>
<td>2.08</td>
<td>1.94</td>
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<td>4: ([1,0,0,-1,T])</td>
<td>2.02</td>
<td>1.98</td>
<td>0.47</td>
<td>13</td>
<td>1.87</td>
<td>1.94</td>
<td>0.32</td>
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<td>1.85</td>
<td>1.99</td>
<td>0.46</td>
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<td>6: ([1,0,1,1,T])</td>
<td>1.74</td>
<td>1.85</td>
<td>0.37</td>
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<td>1.69</td>
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<td>7: ([1,0,1,2,T])</td>
<td>1.92</td>
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<td>14</td>
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<td>8: ([1,0,1,-1,T])</td>
<td>1.86</td>
<td>1.88</td>
<td>0.34</td>
<td>15</td>
<td>1.79</td>
<td>1.87</td>
<td>0.39</td>
<td>22</td>
</tr>
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<td>9: ([1,0,1,-2,T])</td>
<td>1.74</td>
<td>1.74</td>
<td>0.43</td>
<td>14</td>
<td>1.82</td>
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<td>0.40</td>
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<td>10: ([1,0,-1,1,T])</td>
<td>2.00</td>
<td>2.00</td>
<td>0.32</td>
<td>22</td>
<td>1.81</td>
<td>1.94</td>
<td>0.42</td>
<td>18</td>
</tr>
<tr>
<td>11: ([1,0,-2,1,T])</td>
<td>1.97</td>
<td>1.99</td>
<td>0.39</td>
<td>14</td>
<td>2.17</td>
<td>2.14</td>
<td>0.40</td>
<td>18</td>
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<td>1.88</td>
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</tr>
<tr>
<td>13: ([1,1,0,T,0])</td>
<td>1.89</td>
<td>1.88</td>
<td>0.31</td>
<td>20</td>
<td>1.82</td>
<td>1.88</td>
<td>0.39</td>
<td>26</td>
</tr>
<tr>
<td>14: ([1,1,1,1,T])</td>
<td>2.31</td>
<td>2.21</td>
<td>0.41</td>
<td>16</td>
<td>1.75</td>
<td>1.86</td>
<td>0.44</td>
<td>15</td>
</tr>
<tr>
<td>15: ([1,1,-1,1,T])</td>
<td>2.02</td>
<td>2.01</td>
<td>0.30</td>
<td>11</td>
<td>1.87</td>
<td>1.91</td>
<td>0.32</td>
<td>19</td>
</tr>
<tr>
<td>16: ([1,1,-2,1,T])</td>
<td>1.95</td>
<td>1.91</td>
<td>0.33</td>
<td>26</td>
<td>1.98</td>
<td>1.97</td>
<td>0.26</td>
<td>18</td>
</tr>
<tr>
<td>17: ([1,1,-3,1,T])</td>
<td>1.79</td>
<td>1.78</td>
<td>0.25</td>
<td>13</td>
<td>2.00</td>
<td>2.06</td>
<td>0.44</td>
<td>16</td>
</tr>
<tr>
<td>18: ([1,-2,0,T,0])</td>
<td>1.97</td>
<td>2.05</td>
<td>0.33</td>
<td>24</td>
<td>1.91</td>
<td>1.92</td>
<td>0.44</td>
<td>24</td>
</tr>
<tr>
<td>19: ([-1,1,0,1,T])</td>
<td>2.11</td>
<td>2.12</td>
<td>0.40</td>
<td>21</td>
<td>1.71</td>
<td>1.88</td>
<td>0.43</td>
<td>17</td>
</tr>
<tr>
<td>20: ([-1,1,-2,1,T])</td>
<td>1.86</td>
<td>1.92</td>
<td>0.28</td>
<td>23</td>
<td>1.95</td>
<td>1.90</td>
<td>0.36</td>
<td>18</td>
</tr>
<tr>
<td>21: ([-1,1,-3,1,T])</td>
<td>2.07</td>
<td>2.12</td>
<td>0.57</td>
<td>14</td>
<td>1.81</td>
<td>1.81</td>
<td>0.41</td>
<td>19</td>
</tr>
<tr>
<td>All Curves</td>
<td>1.95</td>
<td>1.97</td>
<td>0.37</td>
<td>350</td>
<td>1.85</td>
<td>1.90</td>
<td>0.40</td>
<td>388</td>
</tr>
<tr>
<td>Distinct Curves</td>
<td>1.95</td>
<td>1.97</td>
<td>0.37</td>
<td>335</td>
<td>1.85</td>
<td>1.91</td>
<td>0.40</td>
<td>366</td>
</tr>
</tbody>
</table>
Observe the medians and means of the small conductor set to be larger than those from the large conductor set.

For all curves the Pooled and Unpooled Two-Sample $t$-Procedure give $t$-statistics of 2.5 with over 600 degrees of freedom.

For distinct curves the $t$-statistics is 2.16 (respectively 2.17) with over 600 degrees of freedom (about a 3% chance).

Provides evidence against the null hypothesis (that the means are equal) at the .05 confidence level (though not at the .01 confidence level).
Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$)

1st Normalized Zero above Central Point

Figure 5a: 35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, $\tilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$
Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$) 
1st Normalized Zero above Central Point

Figure 5b: 34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, $\tilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$
Rank 2 Curves: 1st Norm Zero: rank 2 one-param over \(\mathbb{Q}(T)\)

\[\log(\text{cond}) \in [15, 16], \quad t \in [0, 120], \quad \text{median is 1.64.}\]

<table>
<thead>
<tr>
<th>Family</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>log(conductor)</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: ([1,T,0,-3-2T,1])</td>
<td>1.91</td>
<td>0.25</td>
<td>([15.74,16.00])</td>
<td>2</td>
</tr>
<tr>
<td>2: ([1,T,-19,-T-1,0])</td>
<td>1.57</td>
<td>0.36</td>
<td>([15.17,15.63])</td>
<td>4</td>
</tr>
<tr>
<td>3: ([1,T,2,-T-1,0])</td>
<td>1.29</td>
<td>0.25</td>
<td>([15.47,15.47])</td>
<td>1</td>
</tr>
<tr>
<td>4: ([1,T,-16,-T-1,0])</td>
<td>1.75</td>
<td>0.19</td>
<td>([15.07,15.86])</td>
<td>4</td>
</tr>
<tr>
<td>5: ([1,T,13,-T-1,0])</td>
<td>1.53</td>
<td>0.25</td>
<td>([15.08,15.91])</td>
<td>3</td>
</tr>
<tr>
<td>6: ([1,T,-14,-T-1,0])</td>
<td>1.69</td>
<td>0.32</td>
<td>([15.06,15.22])</td>
<td>3</td>
</tr>
<tr>
<td>7: ([1,T,10,-T-1,0])</td>
<td>1.62</td>
<td>0.28</td>
<td>([15.70,15.89])</td>
<td>3</td>
</tr>
<tr>
<td>8: ([0,T,11,-T-1,0])</td>
<td>1.98</td>
<td></td>
<td>([15.87,15.87])</td>
<td>1</td>
</tr>
<tr>
<td>9: ([1,T,-11,-T-1,0])</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10: ([0,T,7,-T-1,0])</td>
<td>1.54</td>
<td>0.17</td>
<td>([15.08,15.90])</td>
<td>7</td>
</tr>
<tr>
<td>11: ([1,T,-8,-T-1,0])</td>
<td>1.58</td>
<td>0.18</td>
<td>([15.23,25.95])</td>
<td>6</td>
</tr>
<tr>
<td>12: ([1,T,19,-T-1,0])</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13: ([0,T,3,-T-1,0])</td>
<td>1.96</td>
<td>0.25</td>
<td>([15.23,15.66])</td>
<td>3</td>
</tr>
<tr>
<td>14: ([0,T,19,-T-1,0])</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15: ([1,T,17,-T-1,0])</td>
<td>1.64</td>
<td>0.23</td>
<td>([15.09,15.98])</td>
<td>4</td>
</tr>
<tr>
<td>16: ([0,T,9,-T-1,0])</td>
<td>1.59</td>
<td>0.29</td>
<td>([15.01,15.85])</td>
<td>5</td>
</tr>
<tr>
<td>17: ([0,T,1,-T-1,0])</td>
<td>1.51</td>
<td></td>
<td>([15.99,15.99])</td>
<td>1</td>
</tr>
<tr>
<td>18: ([1,T,-7,-T-1,0])</td>
<td>1.45</td>
<td>0.23</td>
<td>([15.14,15.43])</td>
<td>4</td>
</tr>
<tr>
<td>19: ([1,T,8,-T-1,0])</td>
<td>1.53</td>
<td>0.24</td>
<td>([15.02,15.89])</td>
<td>10</td>
</tr>
<tr>
<td>20: ([1,T,2,-T-1,0])</td>
<td>1.60</td>
<td></td>
<td>([15.98,15.98])</td>
<td>1</td>
</tr>
<tr>
<td>21: ([0,T,13,-T-1,0])</td>
<td>1.67</td>
<td>0.01</td>
<td>([15.01,15.92])</td>
<td>2</td>
</tr>
</tbody>
</table>

**All Curves** | 1.61 | 0.25 | \([15.01,16.00]\) | 64 |
Function Field Example (with Sal Butt, Chris Hall)

\[ y^2 = x^3 + (t^5 + a_1 t^4 + a_0)x + (t^3 + b_2 t^2 + b_1 t + b_0), \quad a_i, b_i \in \mathbb{F}_5 \]

Figure 6a: Normalized first eigenangle: 719 rank 0 curves.
Function Field Example (with Sal Butt, Chris Hall)

\[ y^2 = x^3 + (t^5 + a_1 t^4 + a_0)x + (t^3 + b_2 t^2 + b_1 t + b_0), \quad a_i, b_i \in \mathbb{F}_5 \]

Figure 6b: Normalized first eigenangle: 978 curves (719 rank 0 curve, 254 rank 2 curves, 5 rank 4 curves).
Jacobi Ensembles
Repulsion or Attraction?

Conductors in \([15, 16]\); first set is rank 0 curves from 14 one-parameter families of rank 0 over \(\mathbb{Q}\); second set rank 2 curves from 21 one-parameter families of rank 0 over \(\mathbb{Q}\). The \(t\)-statistics exceed 6.

<table>
<thead>
<tr>
<th>Family</th>
<th>2nd vs 1st Zero</th>
<th>3rd vs 2nd Zero</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rank 0 Curves</td>
<td>2.16</td>
<td>3.41</td>
<td>863</td>
</tr>
<tr>
<td>Rank 2 Curves</td>
<td>1.93</td>
<td>3.27</td>
<td>701</td>
</tr>
</tbody>
</table>

The repulsion from extra zeros at the central point cannot be entirely explained by only collapsing the first zero to the central point while leaving the other zeros alone. **Can also interpret as attraction.**
Comparison b/w One-Param Families of Different Rank, first normalized zero above the central point.

- First is the 701 rank 2 curves from the 21 one-parameter families of rank 0 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$;

- second is the 64 rank 2 curves from the 21 one-parameter families of rank 2 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$.

<table>
<thead>
<tr>
<th>Family</th>
<th>Median</th>
<th>Mean</th>
<th>Std Dev</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rank 2 Curves (Rank 0 Families)</td>
<td>1.926</td>
<td>1.936</td>
<td>0.388</td>
<td>701</td>
</tr>
<tr>
<td>Rank 2 Curves (Rank 2 Families)</td>
<td>1.642</td>
<td>1.610</td>
<td>0.247</td>
<td>64</td>
</tr>
</tbody>
</table>

- $t$-statistic is 6.60, indicating the means differ.

- The mean of the first normalized zero of rank 2 curves in a family above the central point (for conductors in this range) depends on how we choose the curves.
Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j = \text{imaginary part of } j^{\text{th}} \text{ normalized zero above the central point}$;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

<table>
<thead>
<tr>
<th></th>
<th>863 Rank 0 Curves</th>
<th>701 Rank 2 Curves</th>
<th>t-Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Median</strong></td>
<td>$z_2 - z_1$</td>
<td>1.28</td>
<td>1.30</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td>$z_2 - z_1$</td>
<td>1.30</td>
<td>1.34</td>
</tr>
<tr>
<td><strong>StDev</strong></td>
<td>$z_2 - z_1$</td>
<td>0.49</td>
<td>0.51</td>
</tr>
<tr>
<td><strong>Median</strong></td>
<td>$z_3 - z_2$</td>
<td>1.22</td>
<td>1.19</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td>$z_3 - z_2$</td>
<td>1.24</td>
<td>1.22</td>
</tr>
<tr>
<td><strong>StDev</strong></td>
<td>$z_3 - z_2$</td>
<td>0.52</td>
<td>0.47</td>
</tr>
<tr>
<td><strong>Median</strong></td>
<td>$z_3 - z_1$</td>
<td>2.54</td>
<td>2.56</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td>$z_3 - z_1$</td>
<td>2.55</td>
<td>2.56</td>
</tr>
<tr>
<td><strong>StDev</strong></td>
<td>$z_3 - z_1$</td>
<td>0.52</td>
<td>0.52</td>
</tr>
</tbody>
</table>
Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j = \text{imaginary part of the } j^{th} \text{ norm zero above the central point}$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

<table>
<thead>
<tr>
<th></th>
<th>64 Rank 2 Curves</th>
<th>23 Rank 4 Curves</th>
<th>t-Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Median</strong> $z_2 - z_1$</td>
<td>1.26</td>
<td>1.08</td>
<td>1.35</td>
</tr>
<tr>
<td><strong>Mean</strong> $z_2 - z_1$</td>
<td>1.36</td>
<td>1.14</td>
<td></td>
</tr>
<tr>
<td><strong>StDev</strong> $z_2 - z_1$</td>
<td>0.50</td>
<td>0.35</td>
<td></td>
</tr>
<tr>
<td><strong>Median</strong> $z_3 - z_2$</td>
<td>1.22</td>
<td>1.08</td>
<td></td>
</tr>
<tr>
<td><strong>Mean</strong> $z_3 - z_2$</td>
<td>1.29</td>
<td>1.14</td>
<td></td>
</tr>
<tr>
<td><strong>StDev</strong> $z_3 - z_2$</td>
<td>0.49</td>
<td>0.35</td>
<td></td>
</tr>
<tr>
<td><strong>Median</strong> $z_3 - z_1$</td>
<td>2.66</td>
<td>2.46</td>
<td></td>
</tr>
<tr>
<td><strong>Mean</strong> $z_3 - z_1$</td>
<td>2.65</td>
<td>2.43</td>
<td></td>
</tr>
<tr>
<td><strong>StDev</strong> $z_3 - z_1$</td>
<td>0.44</td>
<td>0.42</td>
<td></td>
</tr>
</tbody>
</table>
Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j = \text{imaginary part of the } j^{th} \text{ norm zero above the central point};$
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

<table>
<thead>
<tr>
<th></th>
<th>701 Rank 2 Curves</th>
<th>64 Rank 2 Curves</th>
<th>t-Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median $z_2 - z_1$</td>
<td>1.30</td>
<td>1.26</td>
<td>0.69</td>
</tr>
<tr>
<td>Mean $z_2 - z_1$</td>
<td>1.34</td>
<td>1.36</td>
<td></td>
</tr>
<tr>
<td>StDev $z_2 - z_1$</td>
<td>0.51</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td>Median $z_3 - z_2$</td>
<td>1.19</td>
<td>1.22</td>
<td>1.39</td>
</tr>
<tr>
<td>Mean $z_3 - z_2$</td>
<td>1.22</td>
<td>1.29</td>
<td></td>
</tr>
<tr>
<td>StDev $z_3 - z_2$</td>
<td>0.47</td>
<td>0.49</td>
<td></td>
</tr>
<tr>
<td>Median $z_3 - z_1$</td>
<td>2.56</td>
<td>2.66</td>
<td>1.93</td>
</tr>
<tr>
<td>Mean $z_3 - z_1$</td>
<td>2.56</td>
<td>2.65</td>
<td></td>
</tr>
<tr>
<td>StDev $z_3 - z_1$</td>
<td>0.52</td>
<td>0.44</td>
<td></td>
</tr>
</tbody>
</table>
New model

The joint PDF of $N$ pairs of eigenvalues $\{e^{i\theta_j}\}_{1 \leq j \leq N}$, taken from random orthogonal matrices having other $r$ fixed eigenvalues at $+1$ is

$$d\mathcal{E}_r(\theta_1, \ldots, \theta_N) = C_{N,r} \prod_{j < k} (\cos \theta_k - \cos \theta_j)^2 \prod_{j} (1 - \cos \theta_j)^r \, d\theta_j.$$ 

This probability measure is well defined for $r \in (-1/2, \infty)$. 

47
Example of Decreasing repulsion: $2 \geq r \geq 0$
Example of Decreasing repulsion: $2 \geq r \geq 0$

\[ r' = 1.9167 \]
Example of Decreasing repulsion: $2 \geq r \geq 0$
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r' = 1.7500$

Figure: 1-level density for the ensemble with $r = 1.7500$. 

Graph showing the 1-level density for the ensemble with $r = 1.7500$. The graph illustrates the behavior of the density as a function of $x$.
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r' = 1.6667$
Example of Decreasing repulsion: \(2 \geq r \geq 0\)

\[ r' = 1.5833 \]
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r' = 1.5000$
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r' = 1.4167$
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r' = 1.3333$
Example of Decreasing repulsion: \( 2 \geq r \geq 0 \)

\[ r' = 1.2500 \]

Figure: 1-level density for the ensemble with \( r = 1.2500 \).
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r' = 1.1667$
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r' = \frac{1.0833}{x}$
Example of Decreasing repulsion: $2 \geq r \geq 0$
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r^* = 0.91667$

![Graph showing the level density for the ensemble with $r^* = 0.91667$. The x-axis represents $x$ ranging from 0 to 5, and the y-axis shows the level density ranging from 0 to 2.]
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r' = 0.83333$

Figure: 1-level density for the ensemble with $r = 10^2 = 12$. 
Example of Decreasing repulsion: \( 2 \geq r \geq 0 \)

\[ r^* = .75000 \]
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r' = .66667$

![Graph showing the level density for the ensemble with $r = .66667$.](image)
Example of Decreasing repulsion: $2 \geq r \geq 0$

$$r' = 0.58333$$

Figure: 1-level density for the ensemble with $r = 0.58333$. 

Graph showing the density distribution with $r' = 0.58333$.
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r = .50000$

$r = .50000$

Figure: 1-level density for the ensemble with $r = 6 = 12$. 
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r' = 0.41667$
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r' = 0.3333$
Example of Decreasing repulsion: $2 \geq r \geq 0$
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r = 0.16667$
Example of Decreasing repulsion: $2 \geq r \geq 0$

$r^* = .83333e-1$
Example of Decreasing repulsion: $2 \geq r \geq 0$
The Condensation Parameter $r$

For simplicity, assume that $\mathcal{E}$ is an even orthogonal family depending on a parameter $T \to \infty$. 
The Condensation Parameter $r$

For simplicity, assume that $\mathcal{E}$ is an even orthogonal family depending on a parameter $T \to \infty$.

- The condensation parameter $r$ will progressively decrease from an initial maximum value $r_0$ to a minimum value $r_\infty = 0$ (resp., $r_\infty = 1$ if $\mathcal{E}$ is an odd orthogonal family.)
For simplicity, assume that $\mathcal{E}$ is an even orthogonal family depending on a parameter $T \to \infty$.

- The condensation parameter $r$ will progressively decrease from an initial maximum value $r_0$ to a minimum value $r_\infty = 0$ (resp., $r_\infty = 1$ if $\mathcal{E}$ is an odd orthogonal family.)

- By suitably decreasing $r$ as $T$ increases, the statistics of eigenvalues in this model match many of the theoretical and experimental features observed in the critical zeros of $\mathcal{E}$:
  - “Repulsion” of eigenvalues away from central point when $r > 0$. (The larger $r$, the more repulsion.)
  - “Independent” model statistics when $r = 0$. 
The Effect of the Parameter $r$

- As $r$ varies from $r_0$ to 0 the “central repulsion” decreases and, at $r = 0$, it disappears completely.

- Increasing $r$ merely tends to shift all the eigenvalues to the right: they are pushed away, but the relative spacings between them are basically unchanged.
Caveat: this bibliography hasn’t been updated for a few years, and could be a little out of date. It is meant to serve as a first reference.


Rizzo, *Average root numbers for a non-constant family of elliptic curves*, preprint.


N. Snaith, *Derivatives of random matrix characteristic polynomials with applications to elliptic curves*, preprint.


