What do you mean?

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Means and averages

Given x and y, the average or mean is the number in between

The number in the middle: ArithmeticMean(x,y) = (x + y) / 2.

There is more than one mean that can be defined!

What properties should a mean have? Assume 0 < x ≤ y.

We want:

* x ≤ mean(x,y) ≤ y. Should be “in between”
* mean(x,x) = x.

Does ArithmeticMean(x,y) = (x+y)/2 satisfy these properties?

Claim It Does!

Since 0 < x ≤ y, we have x + x ≤ x + y ≤ y + y.

So we know 2x ≤ x + y ≤ 2y. Divide everything by 2 and we get

x ≤ (x+y)/2 ≤ y or x ≤ ArithmeticMean(x,y) ≤ y.

We proved the first result!

What about the second? Does ArithmeticMean(x,x) equal x?

Yes! ArithmeticMean(x,x) = (x+x)/2 = 2x / 2 = x.

So the ArithmeticMean(x,y) = (x+y)/2 satisfies our two properties.

We write AM(x,y) = ArithmeticMean(x,y) = (x+y)/2 to save space.

Here’s the question: Is there another choice of mean that satisfies the two properties we wish?

We want:

* x ≤ mean(x,y) ≤ y. Should be “in between”
* mean(x,x) = x.

Try mean(x,y) = Sqrt(x y).

Check: Sqrt(2 \* 8) = Sqrt(16) = 4 and that IS between 2 and 8.

Check: Sqrt(1 \* 100) = Sqrt(100) = 10 and that is between 1 and 100.

Check: Sqrt(1 \* 10) = S

So maybe this is another choice of mean. Maybe it also satisfies the two properties….

Let’s show it does.

First property: Show if 0 < x ≤ y then x ≤ Sqrt(x y) ≤ y.

We know x ≤ y so x x ≤ x y ≤ y y

But x2 ≤ x y ≤ y2. Now take the square-root!

Sqrt(x2) = x and Sqrt(y2) = y, so get x ≤ Sqrt(x y) ≤ y, as claimed!

Second is easier!

We have Sqrt(x x) = Sqrt(x2) = x. We are done!

We call this the GEOMETRIC MEAN.

We write GM(x,y) = Sqrt(x y)

So we have two choices of mean:

AM(x, y) = (x + y) / 2

GM(x, y) = Sqrt(x y)

BOTH have two good properties:

For 0 < x ≤ y both satisfy x ≤ mean(x,y) ≤ y and mean(x,x) = x.

More used to the first.

Try x = 2 and y = 8:

Get AM(2,8) = (2 + 8) / 2 = 10 / 2 = 5

Get GM(2,8) = Sqrt(2 \* 8) = Sqrt(16) = 4

Try x = 3 and y = 12

Then AM(3, 12) = 15/2 = 7.5

And GM(3,12) = Sqrt(36) = 6.

Try x = 1 and y is VERY large….

Then AM(1,y) = (1 + y)/2 which is APPROXIMATELY y/2

But GM(1,y) = Sqrt(y) which is MUCH smaller if y is large.

Note if y is small we would say (1 + y)/2 is approximately .5

CONJECTURE: GM(x,y) ≤ AM(x,y)

PROOF:

Consider: 0 < x ≤ y, what is true about ( Sqrt(x) - Sqrt(y) )2 ? It must be positive…

So 0 ≤ ( Sqrt(x) - Sqrt(y) )2.

Remember FOIL: (a - b)2  = (a - b) (a - b) = a a - a b - b a + b b

First Outside Inside Last

So (a-b)2 = a2 - 2 a b + b2

We are looking at ( Sqrt(x) - Sqrt(y) )2.

0 ≤ ( Sqrt(x) - Sqrt(y) )2 = Sqrt(x)2 - 2 Sqrt(x) Sqrt(y) + Sqrt(y)2.

0 ≤ x - 2 Sqrt(x y) + y

Trying to get AM(x,y) = (x+y)/2 and GM(x,y) = Sqrt(x,y)

2 Sqrt(x,y) ≤ x + y

Sqrt(x,y) ≤ (x+y)/2

GM(x,y) ≤ AM(x,y).

We proved it!

Two things to think about:

What if we had three objects: 0 < x ≤ y ≤ z?

AM(x,y,z) = (x+y+z) / 3

GM(x,y,z) = (x y z)1/3.

Is there another combination?

((x y + y z + x z) / ?? )??

Food for thought: can you find a choice of a and b such that

((xy + yz + zx) / a)b is a mean, so it would satisfy

x ≤ TripleMean(x,y,z) ≤ z and TripleMean(x,x,x) = x

If x = y = z then ((xx + xx + xx) / a)b = (3 x2 / a)b = x for ALL x.

SO b = ½ and a = 3

SO this is our guess….

Try x = 3 and y = 4 and z = 5

TripleMean(3,4,5) = ( (12 + 20 + 15) / 3 )1/2 = (47/3)1/2 is approximately 3.958

This IS a reasonable answer! It is more than 3, less than 5!

Ending on the following:

AM(x,y) = (x+y)/2 GM(x,y) = Sqrt(x y)

Test 1 Get 1 and on Test 2 get 100

AM(1, 100) = (1 + 100)/2 = 50.5

GM(1,100) = Sqrt(1 100) = 10

Log(x y) = Log(x) + Log(y)

So there is a relation between logarithms, AM and GM