What do you mean?

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Means and averages

Given x and y, the average or mean is the number in between

The number in the middle: ArithmeticMean(x,y) = (x + y) / 2.

There is more than one mean that can be defined!

What properties should a mean have? Assume $0 < x \le y$.

We want:

- $x \le \text{mean}(x,y) \le y$. Should be "in between"
- $\text{mean}(x,x) = x$.

Does ArithmeticMean(x,y) = (x+y)/2 satisfy these properties?

Claim It Does!

Since $0 < x \le y$, we have $x + x \le x + y \le y + y$.

So we know $2x \le x + y \le 2y$. Divide everything by 2 and we get

$x \le (x+y)/2 \le y$ or $x \le \text{ArithmeticMean}(x,y) \le y$.

We proved the first result!

What about the second? Does ArithmeticMean(x,x) equal x?

Yes! ArithmeticMean(x,x) = (x+x)/2 = 2x / 2 = x.

So the ArithmeticMean(x,y) = (x+y)/2 satisfies our two properties.

We write AM(x,y) = ArithmeticMean(x,y) = (x+y)/2 to save space.

Here's the question: Is there another choice of mean that satisfies the two properties we wish?

We want:

- $x \leq \text{mean}(x,y) \leq y$. Should be "in between"
- $\text{mean}(x,x) = x$.

Try mean(x,y) = Sqrt(x y).

Check: Sqrt(2 * 8) = Sqrt(16) = 4 and that IS between 2 and 8.

Check: Sqrt(1 * 100) = Sqrt(100) = 10 and that is between 1 and 100.

Check: Sqrt(1 * 10) = S

So maybe this is another choice of mean. Maybe it also satisfies the two properties….

Let's show it does.

First property: Show if $0 < x \leq y$ then $x \leq \text{Sqrt}(x\ y) \leq y$.

We know $x \leq y$ so $x\ x \leq x\ y \leq y\ y$

But $x^2 \leq x\ y \leq y^2$. Now take the square-root!

Sqrt(x^2) = x and Sqrt(y^2) = y, so get $x \leq \text{Sqrt}(x\ y) \leq y$, as claimed!

Second is easier!

We have Sqrt(x x) = Sqrt(x^2) = x. We are done!

We call this the GEOMETRIC MEAN.

We write GM(x,y) = Sqrt(x y)

So we have two choices of mean:

AM(x, y) = (x + y) / 2

GM(x, y) = Sqrt(x y)

BOTH have two good properties:

For $0 < x \le y$ both satisfy $x \le$ mean(x,y) $\le y$ and mean(x,x) = x.

More used to the first.

Try x = 2 and y = 8:

Get AM(2,8) = (2 + 8) / 2 = 10 / 2 = 5

Get GM(2,8) = Sqrt(2 * 8) = Sqrt(16) = 4

Try x = 3 and y = 12

Then AM(3, 12) = 15/2 = 7.5

And GM(3,12) = Sqrt(36) = 6.

Try $x = 1$ and y is VERY large….

Then $AM(1,y) = (1 + y)/2$ which is APPROXIMATELY $y/2$

But $GM(1,y) = Sqrt(y)$ which is MUCH smaller if y is large.

Note if y is small we would say $(1 + y)/2$ is approximately .5

CONJECTURE: $GM(x,y) \leq AM(x,y)$

PROOF:

Consider: $0 < x \leq y$, what is true about $(Sqrt(x) - Sqrt(y))^2$? It must be positive…

So $0 \leq (Sqrt(x) - Sqrt(y))^2$.

Remember FOIL: $(a - b)^2 = (a - b) (a - b) = a a - a b - b a + b b$

First Outside Inside Last

So $(a-b)^2 = a^2 - 2 a b + b^2$

We are looking at $(Sqrt(x) - Sqrt(y))^2$.

$0 \leq (Sqrt(x) - Sqrt(y))^2 = Sqrt(x)^2 - 2 Sqrt(x) Sqrt(y) + Sqrt(y)^2$.

$0 \leq x - 2 Sqrt(x y) + y$

Trying to get $AM(x,y) = (x+y)/2$ and $GM(x,y) = Sqrt(x,y)$

$2 Sqrt(x,y) \leq x + y$

$\text{Sqrt}(x,y) \leq (x+y)/2$

$GM(x,y) \leq AM(x,y)$.

We proved it!

Two things to think about:

What if we had three objects: $0 < x \leq y \leq z$?

$AM(x,y,z) = (x+y+z) / 3$

$GM(x,y,z) = (x\ y\ z)^{1/3}$.

Is there another combination?

$((x\ y\ +\ y\ z\ +\ x\ z) / ??\ )^{??}$

Food for thought: can you find a choice of a and b such that

$((xy + yz + zx) / a)^b$ is a mean, so it would satisfy

$x \leq \text{TripleMean}(x,y,z) \leq z$ and $\text{TripleMean}(x,x,x) = x$

If $x = y = z$ then $((xx + xx + xx) / a)^b = (3\ x^2 / a)^b = x$ for ALL x.

SO $b = ½$ and $a = 3$

SO this is our guess….

Try x = 3 and y = 4 and z = 5

TripleMean(3,4,5) = ((12 + 20 + 15) / 3 $)^{1/2}$ = $(47/3)^{1/2}$ is approximately 3.958

This IS a reasonable answer! It is more than 3, less than 5!

Ending on the following:

AM(x,y) = (x+y)/2   GM(x,y) = Sqrt(x y)

Test 1 Get 1 and on Test 2 get 100

AM(1, 100) = (1 + 100)/2 = 50.5

GM(1,100) = Sqrt(1 100) = 10

Log(x y) = Log(x) + Log(y)

So there is a relation between logarithms, AM and GM