WHAT DO YOUMEAN?!?

Steven Miller, Williams College (sjm1@Williams.edu)

Definitions

Means and averages

- Given x and y, the average or mean is the number in between
- ArithmeticMean(x,y) = (x + y) / 2.
- There is more than one mean that can be defined!
- What properties should a mean have? Assume $0 < x \leq y$.

Desired Properties

We want:

- $x \leq \text{mean}(x,y) \leq y$. Should be “in between”
- $\text{mean}(x,x) = x$.

Does $\text{ArithmeticMean}(x,y) = (x+y)/2$ satisfy these properties?

Desired Properties

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1. $x \leq mean(x,y) \leq y$. Should be "in between"
2. $mean(x,x) = x$.

Does ArithmeticMean(x,y) = (x+y)/2 satisfy these properties?

Proof of (1): Since $0 < x \leq y$, we have $x + x \leq x + y \leq y + y$.
So we know $2x \leq x + y \leq 2y$. Divide everything by 2 and we get
$x \leq (x+y)/2 \leq y$ or $x \leq ArithmeticMean(x,y) \leq y$.
We proved the first result!

Desired Properties

We want:

1. $x \leq mean(x,y) \leq y$. Should be “in between”
2. $mean(x,x) = x$.

Does ArithmeticMean(x,y) = (x+y)/2 satisfy these properties?

Proof of (2):Does ArithmeticMean(x,x) equal x?

Yes! ArithmeticMean(x,x) = (x+x)/2 = 2x / 2 = x.

So the ArithmeticMean(x,y) = (x+y)/2 satisfies our two properties.

We write AM(x,y) = ArithmeticMean(x,y) = (x+y)/2 to save space.

Question

Is there another choice of mean that satisfies the two properties we wish?

We want:

1. $x \le \text{mean}(x,y) \le y$. Should be "in between"
2. $\text{mean}(x,x) = x$.

Thoughts?

Question

Is there another choice of mean that satisfies the two properties we wish?

We want:

1. $x \leq mean(x,y) \leq y$. Should be “in between”
2. $mean(x,x) = x$.

Try mean(x,y) = Sqrt(x y).

- Check: Sqrt(2 * 8) = Sqrt(16) = 4 and that IS between 2 and 8.
- Check: Sqrt(1 * 100) = Sqrt(100) = 10 and that is between 1 and 100.

So maybe this is another choice of mean. Maybe it also satisfies the two properties….

Question

Try mean(x,y) = Sqrt(x,y). Must show

1. x ≤ mean(x,y) ≤ y. Should be "in between"
2. mean(x,x) = x.

First property: Show if 0 < x ≤ y then x ≤ Sqrt(x y) ≤ y.

We know x ≤ y so x x ≤ x y ≤ y y

But $x^2 \leq x\,y \leq y^2$. Now take the square-root!

Sqrt(x^2) = x and Sqrt(y^2) = y, so get x ≤ Sqrt(x y) ≤ y, as claimed!

Question

Try mean(x,y) = Sqrt(x,y). Must show

1. x ≤ mean(x,y) ≤ y. Should be "in between"
2. mean(x,x) = x.

Second is easier!

We have Sqrt(x x) = Sqrt(x^2) = x. We are done!

We call this the GEOMETRIC MEAN. We write GM(x,y) = Sqrt(x y)

Two Means

So we have two choices of mean:

- AM(x, y) = (x + y) / 2
- GM(x, y) = Sqrt(x y)

BOTH have two good properties:

- For $0 < x \leq y$ both satisfy $x \leq \text{mean}(x,y) \leq y$ and mean(x,x) = x.

More used to the first.

Try x = 2 and y = 8:

- Get AM(2,8) = (2 + 8) / 2 = 10 / 2 = 5
- Get GM(2,8) = Sqrt(2 * 8) = Sqrt(16) = 4

Two Means

So we have two choices of mean:

- AM(x, y) = (x + y) / 2
- GM(x, y) = Sqrt(x y)

BOTH have two good properties:

- For $0 < x \le y$ both satisfy $x \le \text{mean}(x,y) \le y$ and mean(x,x) = x.

More used to the first.

Try x = 3 and y = 12

- Then AM(3, 12) = 15/2 = 7.5
- And GM(3,12) = Sqrt(36) = 6.

Two Means

So we have two choices of mean:

- AM(x, y) = (x + y) / 2
- GM(x, y) = Sqrt(x y)

BOTH have two good properties:

- For $0 < x \leq y$ both satisfy $x \leq \text{mean}(x,y) \leq y$ and mean(x,x) = x.

Try x = 1 and y is VERY large….

- Then AM(1,y) = (1 + y)/2 which is APPROXIMATELY y/2
- But GM(1,y) = Sqrt(y) which is MUCH smaller if y is large.
- Note if y is small we would say (1 + y)/2 is approximately .5

CONJECTURE: GM(x,y) ??? AM(x,y)

Two Means

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CONJECTURE: GM(x,y) ≤ AM(x,y)

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PROOF: Consider: $0 < x \leq y$, what is true about (Sqrt(x) - Sqrt(y))^2 ? It must be positive…

- So 0 ≤ (Sqrt(x) - Sqrt(y))^2.

Remember FOIL: $(a - b)^2 = (a - b)(a - b) = a a - a b - b a + b b$: First Outside Inside Last

- So $(a-b)^2 = a^2 - 2 a b + b^2$

We are looking at (Sqrt(x) - Sqrt(y))^2.

- 0 ≤ (Sqrt(x) - Sqrt(y))^2 = Sqrt(x)^2 - 2 Sqrt(x) Sqrt(y) + Sqrt(y)^2.
- 0 ≤ x - 2 Sqrt(x y) + y

Trying to get AM(x,y) = (x+y)/2 and GM(x,y) = Sqrt(x,y)

- 2 Sqrt(x,y) ≤ x + y
- Sqrt(x,y) ≤ (x+y)/2
- GM(x,y) ≤ AM(x,y).

We proved it!

Extensions

What if we had three objects: $0 < x \leq y \leq z$?

- $AM(x,y,z) = (x+y+z) / 3$
- $GM(x,y,z) = (x\, y\, z)^{1/3}$.

Is there another combination?

- $((x\,y + y\,z + x\,z) / ???)^{???}$

Food for thought: can you find a choice of a and b such that

- $((xy + yz + zx) / a)^b$ is a mean, so it would satisfy
- $x \leq \text{TripleMean}(x,y,z) \leq z$ and $\text{TripleMean}(x,x,x) = x$

If $x = y = z$ then $((xx + xx + xx) / a)^b = (3\,x^2 / a)^b = x$ for ALL x.

- SO b = ??? and a = ???

Extensions

What if we had three objects: $0 < x \leq y \leq z$?

- $AM(x,y,z) = (x+y+z) / 3$
- $GM(x,y,z) = (x\ y\ z)^{1/3}$.

Is there another combination? YES

- $((x\,y + y\,z + x\,z) / 3\,)^{1/2}$

If $x = y = z$ then $((xx + xx + xx) / a)^b = (3\,x^2 / a)^b = x$ for ALL x.

- SO $b = ½$ and $a = 3$

SO this is our guess….

- Try $x = 3$ and $y = 4$ and $z = 5$
- $TripleMean(3,4,5) = (\,(12 + 20 + 15) / 3\,)^{1/2} = (47/3)^{1/2}$ is approximately 3.958
- This IS a reasonable answer! It is more than 3, less than 5!

Final Thoughts

AM(x,y) = (x+y)/2   GM(x,y) = Sqrt(x y)

Test 1 Get 1 and on Test 2 get 100

- AM(1, 100) = (1 + 100)/2 = 50.5
- GM(1,100) = Sqrt(1 100) = 10

Recall

- Log(x y) = Log(x) + Log(y)
- So there is a relation between logarithms, AM and GM