Some Results on Low-Lying Zeros of L-Functions

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Introduction

Why study zeros of L-functions?

- Infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from L-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.

Distribution of zeros

- $\zeta(s) \neq 0$ for $\Re\mathfrak{e}(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- GRH: error terms.
- GSH: Chebyshev's bias.
- Analytic rank, adjacent spacings: $h(D)$.

Goals

- Determine correct scale and statistics to study zeros of L-functions.
- See similar behavior in different systems (random matrix theory).
- Discuss the tools and techniques needed to prove the results.
- Talk about some open problems.

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \dots$.

Question: What rules govern the spacings between the t_i?

Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w $n^k\alpha$ mod 1.
- Spacings b/w Zeros of L-functions.

Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.

2. Develop an explicit formula relating what we want to study to something we understand.

3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Classical
Random Matrix Theory

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei (Uranium: 200+ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

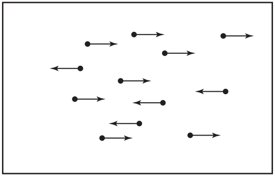
$$H\psi_n = E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Origins of Random Matrix Theory



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Classical Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

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$$\mu_{A,N}(x) \quad = \quad \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

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$$\int_a^b \mu_{A,N}(x)dx = \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N}$$

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$$\text{k}^{\text{th}} \text{ moment} = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.$$

Wigner's Semi-Circle Law

Not most general case, gives flavor.

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) \;=\; \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) \;=\; \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

SKETCH OF PROOF: Correct Scale

$$\mathrm{Trace}(A^2) \;=\; \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\mathrm{Trace}(A^2) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \;\sim\; N^2$$

$$\sum_{i=1}^{N} \lambda_i(A)^2 \;\sim\; N^2$$

Gives $N\mathrm{Ave}(\lambda_i(A)^2) \;\sim\; N^2$ or $\mathrm{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

SKETCH OF PROOF: Averaging Formula

Recall k-th moment of $\mu_{A,N}(x)$ is $\mathrm{Trace}(A^k)/2^k N^{k/2+1}$.

Average k-th moment is

$$\int \cdots \int \frac{\mathrm{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \le j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of k-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).

SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

$$\frac{1}{2^2N^2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ji}^2\cdot p(a_{11})da_{11}\cdots p(a_{NN})da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty}a_{ij}^2p(a_{ij})da_{ij}\quad\cdot\prod_{\substack{(k,l)\neq(i,j)\\ k<l}}\int_{a_{kl}=-\infty}^{\infty}p(a_{kl})da_{kl}\;=\;1.$$

Higher moments involve more advanced combinatorics (Catalan numbers).

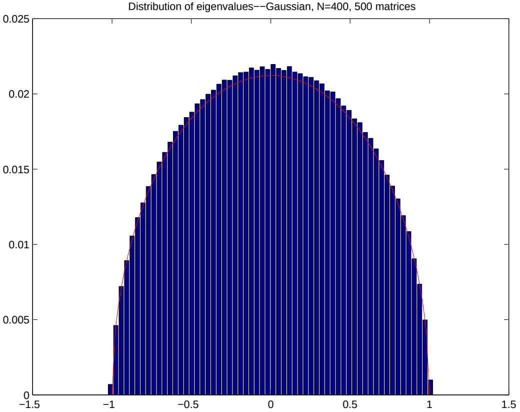
SKETCH OF PROOF: Averaging Formula for Higher Moments

Higher moments involve more advanced combinatorics (Catalan numbers).

$$\frac{1}{2^k N^{k/2+1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} \cdots a_{i_k i_1} \cdot \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

Main term $a_{i_\ell i_{\ell+1}}$'s matched in pairs, not all matchings contribute equally (if did have Gaussian, see in Real Symmetric Palindromic Toeplitz matrices; interesting results for circulant ensembles (joint with Gene Kopp, Murat Kologlu).

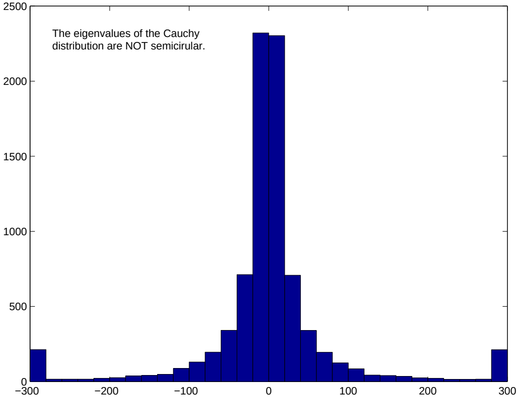
Numerical examples



500 Matrices: Gaussian 400×400

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Numerical examples



Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$

GOE Conjecture

GOE Conjecture:

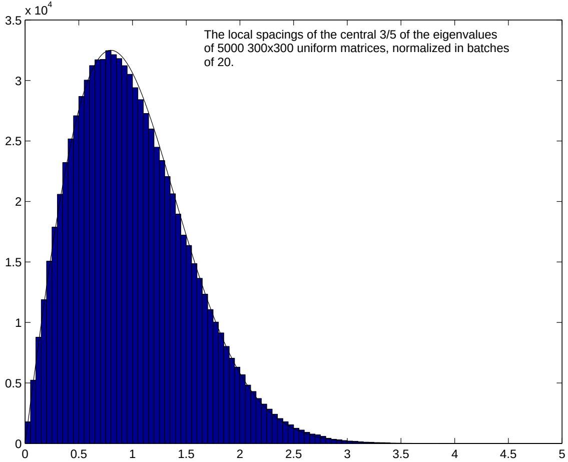
As $N \to \infty$, the probability density of the spacing b/w consecutive normalized eigenvalues approaches a limit independent of p.

Until recently only known if p is a Gaussian.

$$\text{GOE}(x) \;\approx\; \frac{\pi}{2} x e^{-\pi x^2/4}.$$

Numerical Experiment: Uniform Distribution

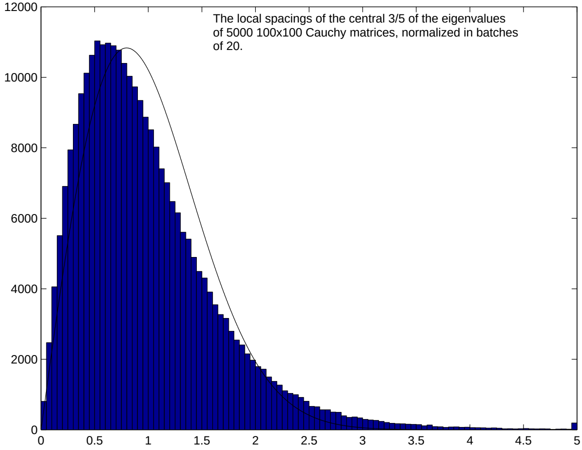
Let $p(x) = \frac{1}{2}$ for $|x| \leq 1$.



5000: 300×300 uniform on $[-1, 1]$

Cauchy Distribution

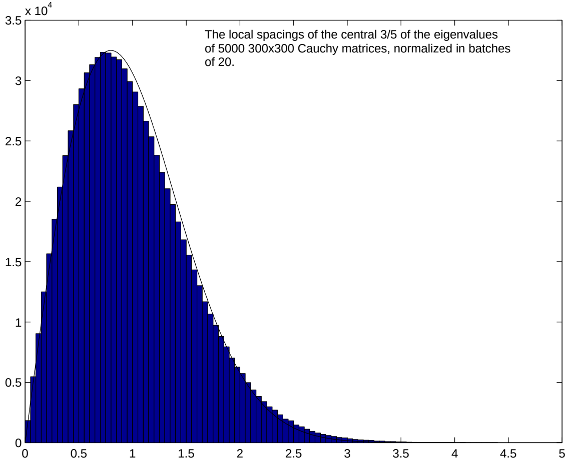
Let $p(x) = \frac{1}{\pi(1+x^2)}$.



5000: 100 $\times$ 100 Cauchy

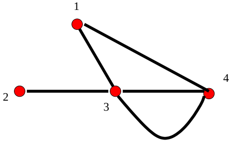
Cauchy Distribution

Let $p(x) = \frac{1}{\pi(1+x^2)}$.



5000: 300×300 Cauchy

Random Graphs



Degree of a vertex $=$ number of edges leaving the vertex.
Adjacency matrix: a_{ij} $=$ number edges b/w Vertex i and Vertex j.

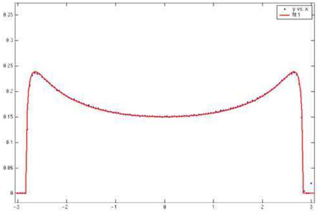
$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

These are Real Symmetric Matrices.

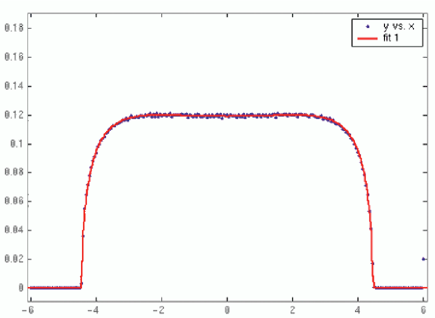
McKay's Law (Kesten Measure) with $d = 3$

Density of Eigenvalues for d-regular graphs

$$f(x) = \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2} & |x| \le 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$



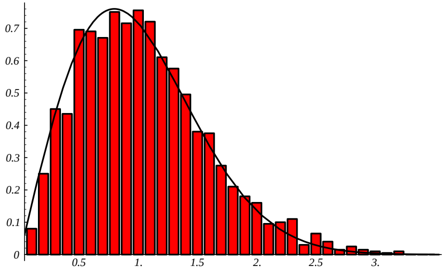
McKay's Law (Kesten Measure) with $d = 6$



Fat Thin: fat enough to average, thin enough to get something different than semi-circle (though as $d \to \infty$ recover semi-circle).

3-Regular Graph with 2000 Vertices: Comparison with the GOE

Spacings between eigenvalues of 3-regular graphs and the GOE:



Introduction to L-Functions

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) = \Lambda_\infty(s,f) L(s,f) = \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

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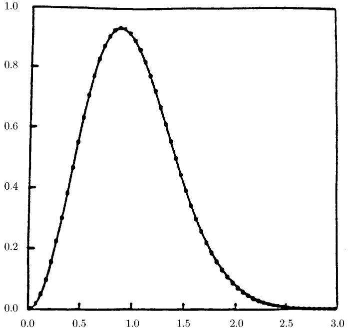
Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko).

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} \;=\; -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).
\end{aligned}
$$

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Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.$$

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Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)p^{-s}ds.$$

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Contour Integration (see Fourier Transform arising):

$$\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)e^{-\sigma\log p}e^{-it\log p}ds.$$

Knowledge of zeros gives info on coefficients.

Explicit Formula: Examples

Riemann Zeta Function: Let $\sum_\rho$ denote the sum over the zeros of $\zeta(s)$ in the critical strip, g an even Schwartz function of compact support and $\phi(r) = \int_{-\infty}^{\infty} g(u)e^{iru}du$. Then

$$\sum_\rho \phi(\gamma_\rho) \;=\; 2\phi\left(\frac{i}{2}\right) - \sum_p \sum_{k=1}^{\infty} \frac{2\log p}{p^{k/2}} g\left(k \log p\right) + \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{1}{iy - \frac{1}{2}} + \frac{\Gamma'(\frac{iy}{2} + \frac{5}{4})}{\Gamma(\frac{iy}{2} + \frac{5}{4}))} - \frac{1}{2}\log \pi\right) \phi(y)\, dy.$$

Explicit Formula: Examples

Dirichlet L-functions: Let h be an even Schwartz function and $L(s,\chi) = \sum_n \chi(n)/n^s$ a Dirichlet L-function from a non-trivial character χ with conductor m and zeros $\rho = \frac{1}{2} + i\gamma_\chi$; if the Generalized Riemann Hypothesis is true then $\gamma \in \mathbb{R}$. Then

$$\sum_\rho h\left(\gamma_\rho \frac{\log(m/\pi)}{2\pi}\right) = \int_{-\infty}^{\infty} h(y)dy$$

$$-2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{h}\left(\frac{\log p}{\log(m/\pi)}\right) \frac{\chi(p)}{p^{1/2}}$$

$$-2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{h}\left(2\frac{\log p}{\log(m/\pi)}\right) \frac{\chi^2(p)}{p} + O\left(\frac{1}{\log m}\right).$$

Explicit Formula: Examples

Cuspidal Newforms: Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) = \widehat{\phi}(0) + \frac{1}{2}\phi(0) - \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} P(f;\phi) + O\left(\frac{\log\log R}{\log R}\right)$$

$$P(f;\phi) = \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.$$

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

n-level correlation

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1} - \alpha_{j_2}, \dots, \alpha_{j_{n-1}} - \alpha_{j_n}\right) \in B, j_i \neq j_k\right\}}{N}$$

(Instead of using a box, can use a smooth test function.)

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko).
2. 2 and 3-correlations of $\zeta(s)$ (Montgomery, Hejhal).
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak).
4. n-level correlations for the classical compact groups (Katz-Sarnak).
5. Insensitive to any finite set of zeros.

Measures of Spacings: n-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) = \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f \gamma_f^{(j_1)}\right) \cdots \phi_n\left(L_f \gamma_f^{(j_n)}\right)$$

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1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.
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Katz-Sarnak Conjecture

For a ‘nice’ family of L-functions, the n-level density depends only on a symmetry group attached to the family.

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathcal{F}_N|^{-1} \sum_{f \in \mathcal{F}_N} \log C_f$). Hope: ϕ a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i \left(\frac{\log C_f}{2\pi} \gamma_E^{(j_i)} \right)$$

$$\to \int \cdots \int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x) dx.$$

Katz-Sarnak Conjecture

As $C_f \to \infty$ the behavior of zeros near $1/2$ agrees with $N \to \infty$ limit of eigenvalues of a classical compact group.

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,\mathrm{SO(even)}}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,\mathrm{SO}}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,\mathrm{SO(odd)}}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1 \end{cases}$$

Correspondences

Similarities between L-Functions and Nuclei:

Zeros	$\longleftrightarrow$	Energy Levels
Schwartz test function	$\longrightarrow$	Neutron
Support of test function	$\longleftrightarrow$	Neutron Energy.

Conjectures and Theorems for Families of Elliptic Curves

1- *and 2-level densities for families of elliptic curves: evidence for the underlying group symmetries*, Compositio Mathematica **140** (2004), 952–992.

http://arxiv.org/pdf/math/0310159.

Tate's Conjecture

Tate's Conjecture for Elliptic Surfaces

Let $\mathcal{E}/\mathbb{Q}$ be an elliptic surface and $L_2(\mathcal{E}, s)$ be the L-series attached to $H^2_{\text{ét}}(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$. Then $L_2(\mathcal{E}, s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies

$$-\mathrm{ord}_{s=2} L_2(\mathcal{E}, s) \ = \ \mathrm{rank}\, NS(\mathcal{E}/\mathbb{Q}),$$

where $NS(\mathcal{E}/\mathbb{Q})$ is the $\mathbb{Q}$-rational part of the Néron-Severi group of $\mathcal{E}$. Further, $L_2(\mathcal{E}, s)$ does not vanish on the line $\mathrm{Re}(s) = 2$.

Conjectures: ABC, Square-Free

ABC Conjecture

Fix $\epsilon > 0$. For coprime positive integers a, b and c with $c = a + b$ and $N(a,b,c) = \prod_{p|abc} p$, $c \ll_\epsilon N(a,b,c)^{1+\epsilon}$.

Square-Free Sieve Conjecture

Fix an irreducible polynomial $f(t)$ of degree at least 4. As $N \to \infty$, the number of $t \in [N, 2N]$ with $D(t)$ divisible by p^2 for some $p > \log N$ is $o(N)$.

Conjectures: Restricted Sign

Restricted Sign Conjecture (for the Family $\mathcal{F}$)

Consider a 1-parameter family $\mathcal{F}$ of elliptic curves. As $N \to \infty$, the signs of the curves E_t are equidistributed for $t \in [N, 2N]$.

Fails: constant $j(t)$ where all curves same sign.
Rizzo:

$$E_t : y^2 = x^3 + tx^2 - (t+3)x + 1, \ \ j(t) = 256(t^2 + 3t + 9),$$

for every $t \in \mathbb{Z}$, E_t has odd functional equation,

$$E_t : y^2 = x^3 + \frac{t}{4}x^2 - \frac{36t^2}{t - 1728}x - \frac{t^3}{t - 1728}, \ \ j(t) = t,$$

as t ranges over $\mathbb{Z}$ in the limit 50.1859% have even and 49.8141% have odd functional equation.

Conjectures: Polynomial Mobius

Polynomial Moebius

Let $f(t)$ be an irreducible polynomial such that no fixed square divides $f(t)$ for all t. Then $\sum_{t=N}^{2N} \mu(f(t)) = o(N)$.

Conjectures: Polynomial Mobius

Helfgott shows the Square-Free Sieve and Polynomial Moebius imply the Restricted Sign conjecture for many families. More precisely, let $M(t)$ be the product of the irreducible polynomials dividing $\Delta(t)$ and not $c_4(t)$.

Theorem

Equidistribution of Sign in a Family Let $\mathcal{F}$ be a one-parameter family with coefficients integer polynomials in $t \in [N, 2N]$. If $j(t)$ and $M(t)$ are non-constant, then the signs of E_t, $t \in [N, 2N]$, are equidistributed as $N \to \infty$. Further, if we restrict to good t, $t \in [N, 2N]$ such that $D(t)$ is good (usually square-free), the signs are still equidistributed in the limit.

Theorem: Preliminaries

Consider a one-parameter family

$$\mathcal{E} : y^2 + a_1(T)xy + a_3(T)y \ = \ x^3 + a_2(T)x^2 + a_4(T)x + a_6(T).$$

Let $a_t(p) = p + 1 - N_p$, where N_p is the number of solutions mod p (including ∞). Define

$$A_{\mathcal{E}}(p) \ := \ \frac{1}{p} \sum_{t(p)} a_t(p).$$

$A_{\mathcal{E}}(p)$ is bounded independent of p (Deligne).

Theorem: Preliminaries

Theorem

Rosen-Silverman (Conjecture of Nagao): For an elliptic surface (a one-parameter family), assume Tate's conjecture. Then

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p \le X} -A_{\mathcal{E}}(p) \log p \ = \ \mathrm{rank}\ \mathcal{E}(\mathbb{Q}(T)).$$

Tate's conjecture is known for rational surfaces: An elliptic surface $y^2 = x^3 + A(T)x + B(T)$ is rational iff one of the following is true:

- $0 < \max\{3\deg A, 2\deg B\} < 12$;
- $3\deg A = 2\deg B = 12$ and $\mathrm{ord}_{T=0} T^{12}\Delta(T^{-1}) = 0$.

Comparing the RMT Models

Theorem: M– '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi} \gamma_{E_t,j}\right)$$

$$= \int \varphi(x)\rho_{\mathcal{G}}(x)dx + r\varphi(0)$$

where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd.} \end{cases}$$

Supports Katz-Sarnak, B-SD, and Independent model in limit.

Data for Elliptic Curve Famillies
Dueñez, Huynh, Keating, Miller and Snaith

Investigations of zeros near the central point of elliptic curve L-functions, Experimental Mathematics **15** (2006), no. 3, 257–279.

http://arxiv.org/pdf/math/0508150.

The lowest eigenvalue of Jacobi Random Matrix Ensembles and Painlevé VI, (with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith), Journal of Physics A: Mathematical and Theoretical **43** (2010) 405204 (27pp).

http://arxiv.org/pdf/1005.1298.

Models for zeros at the central point in families of elliptic curves (with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith), J. Phys. A: Math. Theor. **45** (2012) 115207 (32pp).

http://arxiv.org/pdf/1107.4426.

Comparing the RMT Models

Theorem: M– '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t\in\mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) \\ = \int \varphi(x)\rho_{\mathcal{G}}(x)dx + r\varphi(0)\end{aligned}$$

where

$$\mathcal{G} = \begin{cases} \mathrm{SO} & \text{if half odd} \\ \mathrm{SO(even)} & \text{if all even} \\ \mathrm{SO(odd)} & \text{if all odd.} \end{cases}$$

Supports Katz-Sarnak, B-SD, and Independent model in limit.

Excess rank

One-parameter family, rank $r(\mathcal{E})$ over $\mathbb{Q}(T)$.

For each $t \in \mathbb{Z}$ consider curves E_t.

RMT $\Longrightarrow$ 50% rank $r(\mathcal{E})$, 50% rank $r(\mathcal{E}) + 1$.

For many families, observe

Rank $r(\mathcal{E})$	$=$	32%	Rank $r(\mathcal{E}) + 1$	$=$	48%
rank $r(\mathcal{E}) + 2$	$=$	18%	Rank $r(\mathcal{E}) + 3$	$=$	2%

Problem: small data sets, sub-families, convergence rate log(conductor)?

Excess rank

One-parameter family, rank $r(\mathcal{E})$ over $\mathbb{Q}(T)$.
For each $t \in \mathbb{Z}$ consider curves E_t.
RMT $\Longrightarrow$ 50% rank $r(\mathcal{E})$, 50% rank $r(\mathcal{E}) + 1$.
For many families, observe

$$\begin{array}{lclclcl} \text{Rank } r(\mathcal{E}) & = & 32\% & \quad & \text{Rank } r(\mathcal{E})+1 & = & 48\% \\ \text{rank } r(\mathcal{E})+2 & = & 18\% & \quad & \text{Rank } r(\mathcal{E})+3 & = & 2\% \end{array}$$

Problem: small data sets, sub-families, convergence rate log(conductor)?

Interval	Primes	Twin Primes Pairs
$[1, 10]$	$2, 3, 5, 7$ (40%)	$(3, 5), (5, 7)$ (20%)
$[11, 20]$	$11, 13, 17, 19$ (40%)	$(11, 13), (17, 19)$ (20%)

Small data can be misleading! Remember $\sum_{p \le x} 1/p \sim \log\log x$.

Data on Excess Rank

$$y^2 + a_1xy + a_3y \;=\; x^3 + a_2x^2 + a_4x + a_6$$

Family: a_1 : 0 to 10, rest -10 to 10.
14 Hours, 2,139,291 curves (2,971 singular, 248,478 distinct).

Rank r	=	28.60%	Rank $r+1$	=	47.56%
Rank $r+2$	=	20.97%	Rank $r+3$	=	2.79%
Rank $r+4$	=	.08%			

Data on excess rank (cont)

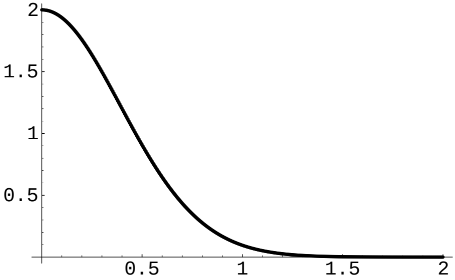
$$y^2 = x^3 + 16Tx + 32$$

Each data set runs over 2000 consecutive t-values.

t-Start	Rk 0	Rk 1	Rk 2	Rk 3	Time (hrs)
-1000	39.4	47.8	12.3	0.6	<1
1000	38.4	47.3	13.6	0.6	<1
4000	37.4	47.8	13.7	1.1	1
8000	37.3	48.8	12.9	1.0	2.5
24000	35.1	50.1	13.9	0.8	6.8
50000	36.7	48.3	13.8	1.2	51.8

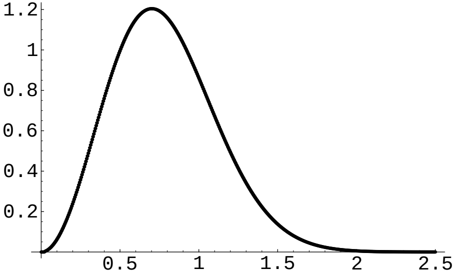
Final conductors $\sim 10^{11}$, small on log scale.

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(even)

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(odd)

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

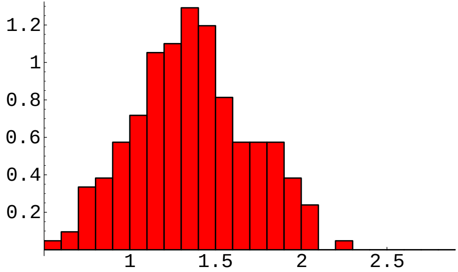


Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

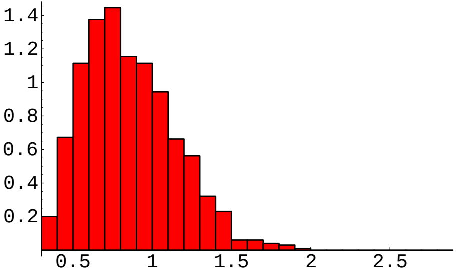


Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$)
1st Normalized Zero above Central Point

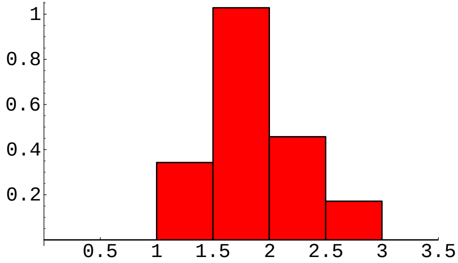


Figure 5*a*: 35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$) 1st Normalized Zero above Central Point

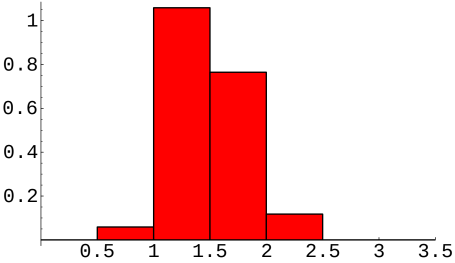


Figure 5b: 34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- z_j = imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	701 Rank 2 Curves	t-Statistic
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2 Curves	23 Rank 4 Curves	t-Statistic
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	**64 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

Summary of Data

- The repulsion of the low-lying zeros increased with increasing rank, and was present even for rank 0 curves.
- As the conductors increased, the repulsion decreased.
- Statistical tests failed to reject the hypothesis that, on average, the first three zeros were all repelled equally (i. e., shifted by the same amount).

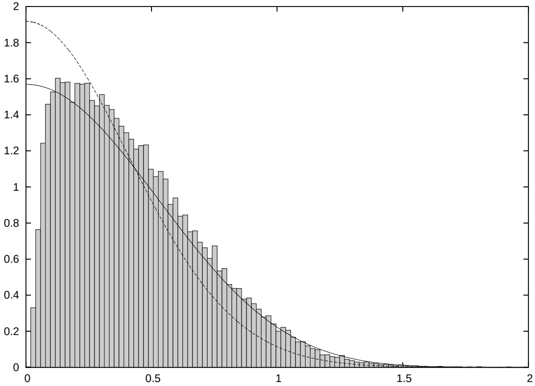
New Model for Finite Conductors

- Replace conductor N with $N_{\text{effective}}$.
  - ◇ Arithmetic info, predict with L-function Ratios Conj.
  - ◇ Do the number theory computation.
- Excised Orthogonal Ensembles.
  - ◇ $L(1/2, E)$ discretized.
  - ◇ Study matrices in $\mathrm{SO}(2N_{eff})$ with $|\Lambda_A(1)| \geq ce^N$.
- Painlevé VI differential equation solver.
  - ◇ Use explicit formulas for densities of Jacobi ensembles.
  - ◇ Key input: Selberg-Aomoto integral for initial conditions.

Open Problem:

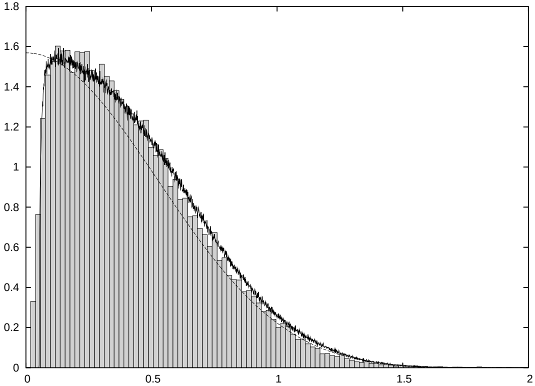
Generalize to other families (ongoing with Nathan Ryan).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with $N_{\rm eff}$ (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): N_{eff} = 2 (solid) with discretisation, and $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

Lower order terms

Variation in the number of points on elliptic curves and applications to excess rank, C. R. Math. Rep. Acad. Sci. Canada **27** (2005), no. 4, 111–120.

http://arxiv.org/pdf/math/0506461v2.pdf.

Lower order terms in the 1-level density for families of holomorphic cuspidal newforms, Acta Arithmetica **137** (2009), 51–98.

http://arxiv.org/pdf/0704.0924.pdf.

Lower Order Terms

Convolve families of elliptic curves with ranks r_1 and r_2: see lower order term of size $r_1 r_2$ (over logarithms).

Difficulty is isolating that from other errors (often of size $\log\log R / \log R$). Study weighted moments

$$\begin{aligned} A_{r,\mathcal{F}}(p) &:= \frac{1}{W_R(\mathcal{F})} \sum_{\substack{f\in\mathcal{F} \\ f\in S(p)}} w_R(f)\lambda_f(p)^r \\ A'_{r,\mathcal{F}}(p) &:= \frac{1}{W_R(\mathcal{F})} \sum_{\substack{f\in\mathcal{F} \\ f\notin S(p)}} w_R(f)\lambda_f(p)^r \\ S(p) &:= \{f \in \mathcal{F} : p \nmid N_f\}. \end{aligned}$$

Main difficulty in 1-level density is evaluating

$$S(\mathcal{F}) = -2\sum_p \sum_{m=1}^{\infty} \frac{1}{W_R(\mathcal{F})} \sum_{f\in\mathcal{F}} w_R(f) \frac{\alpha_f(p)^m + \beta_f(p)^m}{p^{m/2}} \frac{\log p}{\log R}\, \widehat{\phi}\left(m \frac{\log p}{\log R}\right).$$

Fourier Coefficient Expansion

$$\begin{aligned}
S(\mathcal{F}) \ &= \ -2\sum_p \sum_{m=1}^{\infty} \frac{A'_{m,\mathcal{F}}(p)}{p^{m/2}} \frac{\log p}{\log R}\, \widehat{\phi}\left(m\frac{\log p}{\log R}\right) \\
&\quad -2\widehat{\phi}(0)\sum_p \frac{2A_{0,\mathcal{F}}(p)\log p}{p(p+1)\log R} \ + \ 2\sum_p \frac{2A_{0,\mathcal{F}}(p)\log p}{p\log R}\, \widehat{\phi}\left(2\frac{\log p}{\log R}\right) \\
&\quad -2\sum_p \frac{A_{1,\mathcal{F}}(p)}{p^{1/2}}\frac{\log p}{\log R}\, \widehat{\phi}\left(\frac{\log p}{\log R}\right) + 2\widehat{\phi}(0)\frac{A_{1,\mathcal{F}}(p)(3p+1)}{p^{1/2}(p+1)^2}\frac{\log p}{\log R} \\
&\quad -2\sum_p \frac{A_{2,\mathcal{F}}(p)\log p}{p\log R}\, \widehat{\phi}\left(2\frac{\log p}{\log R}\right) + 2\widehat{\phi}(0)\sum_p \frac{A_{2,\mathcal{F}}(p)(4p^2+3p+1)\log p}{p(p+1)^3\log R} \\
&\quad -2\widehat{\phi}(0)\sum_p\sum_{r=3}^{\infty} \frac{A_{r,\mathcal{F}}(p)p^{r/2}(p-1)\log p}{(p+1)^{r+1}\log R} \ + \ O\left(\frac{1}{\log^3 R}\right) \\
&= \ S_{A'}(\mathcal{F}) + S_0(\mathcal{F}) + S_1(\mathcal{F}) + S_2(\mathcal{F}) + S_A(\mathcal{F}) + O\left(\frac{1}{\log^3 R}\right).
\end{aligned}$$

Letting $\widetilde{A}_{\mathcal{F}}(p) \ := \ \frac{1}{W_R(\mathcal{F})}\sum_{f\in S(p)} w_R(f)\frac{\lambda_f(p)^3}{p+1-\lambda_f(p)\sqrt{p}}$, by the geometric series formula we may replace $S_A(\mathcal{F})$ with $S_{\tilde{A}}(\mathcal{F})$, where

$$S_{\tilde{A}}(\mathcal{F}) \ = \ -2\widehat{\phi}(0)\sum_p \frac{\widetilde{A}_{\mathcal{F}}(p)p^{3/2}(p-1)\log p}{(p+1)^3\log R}.$$

Family Dependent Lower Order Terms: Miller '09

$\mathcal{F}_{k,N}$ the family of even weight k and prime level N cuspidal newforms, or just the forms with even (or odd) functional equation.

Up to $O(\log^{-3} R)$, as $N \to \infty$ for test functions ϕ with $\text{supp}(\widehat{\phi}) \subset (-4/3, 4/3)$ the (non-conductor) lower order term is

$$-1.33258 \cdot 2\widehat{\phi}(0)/\log R.$$

Note the lower order corrections are independent of the distribution of the signs of the functional equations.

Family Dependent Lower Order Terms: Miller '09

CM example, with or without forced torsion: $y^2 = x^3 + B(6T+1)^\kappa$ over $\mathbb{Q}(T)$, with $B \in \{1,2,3,6\}$ and $\kappa \in \{1,2\}$.

CM, sieve to $(6T+1)$ is $(6/\kappa)$-power free. If $\kappa = 1$ then all values of B the same, if $\kappa = 2$ the four values of B have different lower order corrections; in particular, if $B = 1$ then there is a forced torsion point of order three, $(0, 6T+1)$.

Up to errors of size $O(\log^{-3} R)$, the (non-conductor) lower order terms are approximately

$$\begin{aligned}
B = 1, \kappa = 1: &\quad -2.124 \cdot 2\widehat{\phi}(0)/\log R,\\
B = 1, \kappa = 2: &\quad -2.201 \cdot 2\widehat{\phi}(0)/\log R,\\
B = 2, \kappa = 2: &\quad -2.347 \cdot 2\widehat{\phi}(0)/\log R\\
B = 3, \kappa = 2: &\quad -1.921 \cdot 2\widehat{\phi}(0)/\log R\\
B = 6, \kappa = 2: &\quad -2.042 \cdot 2\widehat{\phi}(0)/\log R.
\end{aligned}$$

Family Dependent Lower Order Terms: Miller '09

CM example, with or without rank:
$y^2 = x^3 - B(36T+6)(36T+5)x$ over $\mathbb{Q}(T)$, with $B \in \{1,2\}$. If $B = 1$ the family has rank 1, while if $B = 2$ the family has rank 0.

Sieve to $(36T+6)(36T+5)$ is cube-free. Most important difference between these two families is the contribution from the $S_{\widetilde{\mathcal{A}}}(\mathcal{F})$ terms, where the $B = 1$ family is approximately $-.11 \cdot 2\widehat{\phi}(0)/\log R$, while the $B = 2$ family is approximately $.63 \cdot 2\widehat{\phi}(0)/\log R$.

This large difference is due to biases of size $-r$ in the Fourier coefficients $a_t(p)$ in a one-parameter family of rank r over $\mathbb{Q}(T)$.

Main term of the average moments of the p^{th} Fourier coefficients are given by the complex multiplication analogue of Sato-Tate in the limit, for each p there are lower order correction terms which depend on the rank.

Family Dependent Lower Order Terms: Miller '09

Non-CM Example: $y^2 = x^3 - 3x + 12T$ over $\mathbb{Q}(T)$. Up to $O(\log^{-3} R)$, the (non-conductor) lower order correction is approximately

$$-2.703 \cdot 2\widehat{\phi}(0)/\log R,$$

which is very different than the family of weight 2 cuspidal newforms of prime level N.

Explicit calculations

Let $n_{3,2,p}$ equal the number of cube roots of 2 modulo p, and set $c_0(p) = \left[\left(\frac{-3}{p}\right) + \left(\frac{3}{p}\right)\right] p$, $c_1(p) = \left[\sum_{x \bmod p} \left(\frac{x^3-x}{p}\right)\right]^2$, $c_{\frac{3}{2}}(p) = p \sum_{x(p)} \left(\frac{4x^3+1}{p}\right)$.

Family	$A_{1,\mathcal{E}}(p)$	$A_{2,\mathcal{E}}(p)$
$y^2 = x^3 + Sx + T$	0	$p^3 - p^2$
$y^2 = x^3 + 2^4(-3)^3(9T+1)^2$	0	$\begin{cases} 2p^2-2p & p \equiv 2 \bmod 3 \\ 0 & p \equiv 1 \bmod 3 \end{cases}$
$y^2 = x^3 \pm 4(4T+2)x$	0	$\begin{cases} 2p^2-2p & p \equiv 1 \bmod 3 \\ 0 & p \equiv 3 \bmod 3 \end{cases}$
$y^2 = x^3 + (T+1)x^2 + Tx$	0	$p^2 - 2p - 1$
$y^2 = x^3 + x^2 + 2T + 1$	0	$p^2 - 2p - \left(\frac{-3}{p}\right)$
$y^2 = x^3 + Tx^2 + 1$	$-p$	$p^2 - n_{3,2,p}p - 1 + c_{\frac{3}{2}}(p)$
$y^2 = x^3 - T^2x + T^2$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 - T^2x + T^4$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 + Tx^2 - (T+3)x + 1$	$-2c_{p,1;4}p$	$p^2 - 4c_{p,1;6}p - 1$

where $c_{p,a;m} = 1$ if $p \equiv a \bmod m$ and otherwise is 0.

Explicit calculations

The first family is the family of all elliptic curves; it is a two parameter family and we expect the main term of its second moment to be p^3.

Note that except for our family $y^2 = x^3 + Tx^2 + 1$, all the families $\mathcal{E}$ have $A_{2,\mathcal{E}}(p) = p^2 - h(p)p + O(1)$, where $h(p)$ is non-negative. Further, many of the families have $h(p) = m_{\mathcal{E}} > 0$.

Note $c_1(p)$ is the square of the coefficients from an elliptic curve with complex multiplication. It is non-negative and of size p for $p \not\equiv 3 \bmod 4$, and zero for $p \equiv 1 \bmod 4$ (send $x \mapsto -x \bmod p$ and note $\left(\frac{-1}{p}\right) = -1$).

It is somewhat remarkable that all these families have a correction to the main term in Michel's theorem in the same direction, and we analyze the consequence this has on the average rank. For our family which has a $p^{3/2}$ term, note that on average this term is zero and the p term is negative.

Lower order terms and average rank

$$\frac{1}{N}\sum_{t=N}^{2N}\sum_{\gamma_t}\phi\left(\gamma_t\frac{\log R}{2\pi}\right) = \widehat{\phi}(0)+\phi(0)-\frac{2}{N}\sum_{t=N}^{2N}\sum_{p}\frac{\log p}{\log R}\frac{1}{p}\widehat{\phi}\left(\frac{\log p}{\log R}\right)a_t(p)$$
$$-\frac{2}{N}\sum_{t=N}^{2N}\sum_{p}\frac{\log p}{\log R}\frac{1}{p^2}\widehat{\phi}\left(\frac{2\log p}{\log R}\right)a_t(p)^2+O\left(\frac{\log\log R}{\log R}\right).$$

If ϕ is non-negative, we obtain a bound for the average rank in the family by restricting the sum to be only over zeros at the central point. The error $O\left(\frac{\log\log R}{\log R}\right)$ comes from trivial estimation and ignores probable cancellation, and we expect $O\left(\frac{1}{\log R}\right)$ or smaller to be the correct magnitude. For most families $\log R \sim \log N^a$ for some integer a.

Lower order terms and average rank (cont)

The main term of the first and second moments of the $a_t(p)$ give $r\phi(0)$ and $-\frac{1}{2}\phi(0)$.

Assume the second moment of $a_t(p)^2$ is $p^2 - m_{\mathcal{E}}p + O(1)$, $m_{\mathcal{E}} > 0$.

We have already handled the contribution from p^2, and $-m_{\mathcal{E}}p$ contributes

$$\begin{aligned} S_2 &\sim \frac{-2}{N}\sum_p \frac{\log p}{\log R}\widehat{\phi}\left(2\frac{\log p}{\log R}\right)\frac{1}{p^2}\frac{N}{p}(-m_{\mathcal{E}}p) \\ &= \frac{2m_{\mathcal{E}}}{\log R}\sum_p \widehat{\phi}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p^2}. \end{aligned}$$

Thus there is a contribution of size $\frac{1}{\log R}$.

Lower order terms and average rank (cont)

A good choice of test functions (see Appendix A of [ILS]) is the Fourier pair

$$\phi(x) \ = \ \frac{\sin^2(2\pi\frac{\sigma}{2}x)}{(2\pi x)^2}, \quad \widehat{\phi}(u) \ = \ \begin{cases} \frac{\sigma-|u|}{4} & \text{if } |u| \le \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Note $\phi(0) = \frac{\sigma^2}{4}$, $\widehat{\phi}(0) = \frac{\sigma}{4} = \frac{\phi(0)}{\sigma}$, and evaluating the prime sum gives

$$S_2 \quad \sim \quad \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2 \log R}\right) \frac{m_{\mathcal{E}}}{\log R}\ \phi(0).$$

Lower order terms and average rank (cont)

Let r_t denote the number of zeros of E_t at the central point (i.e., the analytic rank). Then up to our $O\left(\frac{\log\log R}{\log R}\right)$ errors (which we think should be smaller), we have

$$\begin{aligned}
\frac{1}{N}\sum_{t=N}^{2N} r_t\phi(0) &\le \frac{\phi(0)}{\sigma} + \left(r+\frac{1}{2}\right)\phi(0) + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathcal{E}}}{\log R}\phi(0) \\
\text{Ave Rank}_{[N,2N]}(\mathcal{E}) &\le \frac{1}{\sigma} + r + \frac{1}{2} + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathcal{E}}}{\log R}.
\end{aligned}$$

$\sigma = 1$, $m_{\mathcal{E}} = 1$: for conductors of size 10^{12}, the average rank is bounded by $1 + r + \frac{1}{2} + .03 = r + \frac{1}{2} + 1.03$. This is significantly higher than Fermigier's observed $r + \frac{1}{2} + .40$.

$\sigma = 2$: lower order correction contributes .02 for conductors of size 10^{12}, the average rank bounded by $\frac{1}{2} + r + \frac{1}{2} + .02 = r + \frac{1}{2} + .52$. Now in the ballpark of Fermigier's bound (already there without the potential correction term!).

Open Questions and References

Open Questions: Low-lying zeros of *L*-functions

- Generalize excised ensembles for higher weight GL_2 families where expect different discretizations.

- Obtain better estimates on vanishing at the central point by finding optimal test functions for the second and higher moment expansions.

- Further explore *L*-function Ratios Conjecture to predict lower order terms in families, compute these terms on number theory side.

See Dueñez-Huynh-Keating-Miller-Snaith, Miller, and the Ratios papers.

Publications: Random Matrix Theory

1. *Distribution of eigenvalues for the ensemble of real symmetric Toeplitz matrices* (with Christopher Hammond), Journal of Theoretical Probability **18** (2005), no. 3, 537–566. http://arxiv.org/abs/math/0312215
2. *Distribution of eigenvalues of real symmetric palindromic Toeplitz matrices and circulant matrices* (with Adam Massey and John Sinsheimer), Journal of Theoretical Probability **20** (2007), no. 3, 637–662. http://arxiv.org/abs/math/0512146
3. *The distribution of the second largest eigenvalue in families of random regular graphs* (with Tim Novikoff and Anthony Sabelli), Experimental Mathematics **17** (2008), no. 2, 231–244. http://arxiv.org/abs/math/0611649
4. *Nuclei, Primes and the Random Matrix Connection* (with Frank W. K. Firk), Symmetry **1** (2009), 64–105; doi:10.3390/sym1010064. http://arxiv.org/abs/0909.4914
5. *Distribution of eigenvalues for highly palindromic real symmetric Toeplitz matrices* (with Steven Jackson and Thuy Pham), Journal of Theoretical Probability **25** (2012), 464–495. http://arxiv.org/abs/1003.2010
6. *The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices* (with Murat Koloğlu, Gene S. Kopp, Frederick W. Strauch and Wentao Xiong), Journal of Theoretical Probability **26** (2013), no. 4, 1020–1060. http://arxiv.org/abs/1008.4812
7. *Distribution of eigenvalues of weighted, structured matrix ensembles* (with Olivia Beckwith, Karen Shen), submitted December 2011 to the Journal of Theoretical Probability, revised September 2012. http://arxiv.org/abs/1112.3719 .
8. *The expected eigenvalue distribution of large, weighted d-regular graphs* (with Leo Goldmahker, Cap Khoury and Kesinee Ninsuwan), preprint.

Publications: L-Functions

1. *The low lying zeros of a $GL(4)$ and a $GL(6)$ family of L-functions* (with Eduardo Dueñez), Compositio Mathematica **142** (2006), no. 6, 1403–1425. http://arxiv.org/abs/math/0506462
2. *Low lying zeros of L–functions with orthogonal symmetry* (with Christopher Hughes), Duke Mathematical Journal **136** (2007), no. 1, 115–172. http://arxiv.org/abs/math/0507450
3. *Lower order terms in the 1-level density for families of holomorphic cuspidal newforms*, Acta Arithmetica **137** (2009), 51–98. http://arxiv.org/abs/0704.0924
4. *The effect of convolving families of L-functions on the underlying group symmetries* (with Eduardo Dueñez), Proceedings of the London Mathematical Society, 2009; doi: 10.1112/plms/pdp018. http://arxiv.org/pdf/math/0607688.pdf
5. *Low-lying zeros of number field L-functions* (with Ryan Peckner), Journal of Number Theory **132** (2012), 2866–2891. http://arxiv.org/abs/1003.5336
6. *The low-lying zeros of level 1 Maass forms* (with Levent Alpoge), preprint 2013. http://arxiv.org/abs/1301.5702
7. *The n-level density of zeros of quadratic Dirichlet L-functions* (with Jake Levinson), submitted September 2012 to Acta Arithmetica. http://arxiv.org/abs/1208.0930
8. *Moment Formulas for Ensembles of Classical Compact Groups* (with Geoffrey Iyer and Nicholas Triantafillou), preprint 2013.

Publications: Elliptic Curves

1. 1- *and* 2*-level densities for families of elliptic curves: evidence for the underlying group symmetries*, Compositio Mathematica **140** (2004), 952–992. http://arxiv.org/pdf/math/0310159

2. *Variation in the number of points on elliptic curves and applications to excess rank*, C. R. Math. Rep. Acad. Sci. Canada **27** (2005), no. 4, 111–120. http://arxiv.org/abs/math/0506461

3. *Investigations of zeros near the central point of elliptic curve L-functions*, Experimental Mathematics **15** (2006), no. 3, 257–279. http://arxiv.org/pdf/math/0508150

4. *Constructing one-parameter families of elliptic curves over* $\mathbb{Q}(T)$ *with moderate rank* (with Scott Arms and Álvaro Lozano-Robledo), Journal of Number Theory **123** (2007), no. 2, 388–402. http://arxiv.org/abs/math/0406579

5. *Towards an 'average' version of the Birch and Swinnerton-Dyer Conjecture* (with John Goes), Journal of Number Theory **130** (2010), no. 10, 2341–2358. http://arxiv.org/abs/0911.2871

6. *The lowest eigenvalue of Jacobi Random Matrix Ensembles and Painlevé VI*, (with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith), Journal of Physics A: Mathematical and Theoretical **43** (2010) 405204 (27pp). http://arxiv.org/pdf/1005.1298

7. *Models for zeros at the central point in families of elliptic curves* (with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith), J. Phys. A: Math. Theor. **45** (2012) 115207 (32pp). http://arxiv.org/pdf/1107.4426

8. *Effective equidistribution and the Sato-Tate law for families of elliptic curves* (with M. Ram Murty), Journal of Number Theory **131** (2011), no. 1, 25–44. http://arxiv.org/abs/1004.2753

9. *Moments of the rank of elliptic curves* (with Siman Wong), Canad. J. of Math. **64** (2012), no. 1, 151–182. http://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/mwMomentsRanksEC81

Publications: L-Function Ratio Conjecture

1. *A symplectic test of the L-Functions Ratios Conjecture*, Int Math Res Notices (2008) Vol. 2008, article ID rnm146, 36 pages, doi:10.1093/imrn/rnm146. http://arxiv.org/abs/0704.0927
2. *An orthogonal test of the L-Functions Ratios Conjecture*, Proceedings of the London Mathematical Society 2009, doi:10.1112/plms/pdp009. http://arxiv.org/abs/0805.4208
3. *A unitary test of the L-functions Ratios Conjecture* (with John Goes, Steven Jackson, David Montague, Kesinee Ninsuwan, Ryan Peckner and Thuy Pham), Journal of Number Theory **130** (2010), no. 10, 2238–2258. http://arxiv.org/abs/0909.4916
4. *An Orthogonal Test of the L-functions Ratios Conjecture, II* (with David Montague), Acta Arith. **146** (2011), 53–90. http://arxiv.org/abs/0911.1830
5. *An elliptic curve family test of the Ratios Conjecture* (with Duc Khiem Huynh and Ralph Morrison), Journal of Number Theory **131** (2011), 1117–1147. http://arxiv.org/abs/1011.3298
6. *Surpassing the Ratios Conjecture in the 1-level density of Dirichlet L-functions* (with Daniel Fiorilli). submitted September 2012 to Proceedings of the London Mathematical Society. http://arxiv.org/abs/1111.3896

Thank you!