Finite conductor models for zeros near the central point of elliptic curve L-functions

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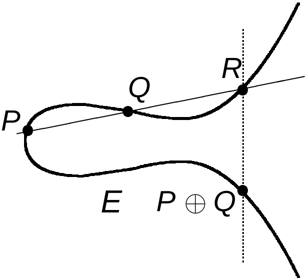
Outline

- Review elliptic curves.
- Results for large conductors.
- Data for small conductors.
- Reconciling theory and data.

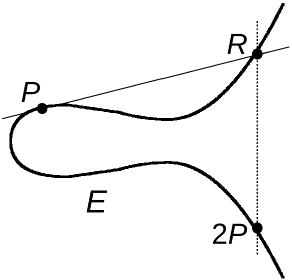
Elliptic Curves

Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.



Addition of distinct points P and Q



Adding a point P to itself

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

General L-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p\left(s,f\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f)L(s,f) \;=\; \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (GRH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Elliptic curve L-function

$E : y^2 = x^3 + ax + b$, associate L-function

$$L(s,E) \;=\; \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x,y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

Birch and Swinnerton-Dyer Conjecture

Rank of group of rational solutions equals order of vanishing of $L(s,E)$ at $s = 1/2$.

One parameter family

$$\mathcal{E} : y^2 = x^3 + A(T)x + B(T), A(T), B(T) \in \mathbb{Z}[T].$$

Silverman's Specialization Theorem

Assume (geometric) rank of $\mathcal{E}/\mathbb{Q}(T)$ is r. Then for all $t \in \mathbb{Z}$ sufficiently large, each $E_t : y^2 = x^3 + A(t)x + B(t)$ has (geometric) rank at least r.

Average rank conjecture

For a generic one-parameter family of rank r over $\mathbb{Q}(T)$, expect in the limit half the specialized curves have rank r and half have rank $r + 1$.

Classical Random Matrix Theory

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at t_1, t_2, $t_3, \dots$.

Question: What rules govern the spacings between the t_i?

Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w $n^k\alpha$ mod 1.
- Spacings b/w Zeros of L-functions.

Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.

2. Develop an explicit formula relating what we want to study to something we understand.

3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

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Heavy nuclei (Uranium: 200$+$ protons / neutrons) worse!

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Fundamental Equation:

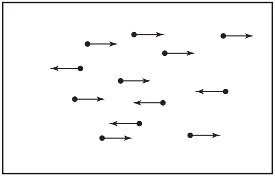
$$H\psi_n \;=\; E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Origins (continued)



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0)$.

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To each A, attach a probability measure:

$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

$$\int_a^b \mu_{A,N}(x)dx = \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N}$$

$$\text{k}^{\text{th}} \text{ moment} = \frac{\sum_{i=1}^{N} \lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}}.$$

Eigenvalue Trace Lemma

Want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

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Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

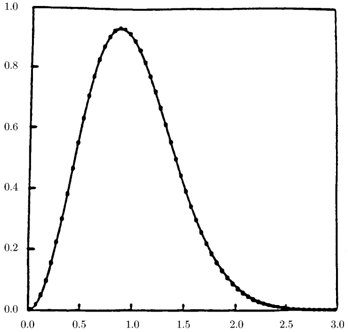
$$\mathrm{Trace}(A^k) \;=\; \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) \;=\; \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

Limiting Behavior

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko) versus RMT prediction.

1-Level Density

L-function $L(s, f)$: by RH non-trivial zeros $\frac{1}{2} + i\gamma_{f,j}$.
C_f: analytic conductor.
$\varphi(x)$: compactly supported even Schwartz function.

$$D_{1,f}(\varphi) \quad = \quad \sum_j \varphi\left(\frac{\log C_f}{2\pi}\gamma_{f,j}\right)$$

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Katz-Sarnak Conjecture:

$$\begin{aligned} D_{1,\mathcal{F}}(\varphi) = \lim_{N\to\infty}\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{1,f}(\varphi) \quad &= \quad \int \varphi(x)\rho_{G(\mathcal{F})}(x)dx \\ &= \quad \int \widehat{\varphi}(u)\widehat{\rho}_{G(\mathcal{F})}(u)du. \end{aligned}$$

Comparing with Random Matrix Theory

Theorem: M– '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t\in\mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) = \int \varphi(x)\rho_{\mathcal{G}}(x)dx + r\varphi(0)$$

where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

Confirm Katz-Sarnak, B-SD predictions for small support.

Supports Independent and not Interaction model in the limit.

Sketch of Proof

- Explicit Formula: Relates sums over zeros to sums over primes.
- Averaging Formulas: Orthogonality of characters, Petersson formula.
- Control of conductors: Monotone.

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) \ = \ -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \ \cdot \ p^{-s}}{1-p^{-s}} \ = \ \sum_p \frac{\log p}{p^s} \ + \ \mathrm{Good}(s).
\end{aligned}
$$

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Contour Integration:

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)p^{-s}ds.
$$

Explicit Formula (Contour Integration)

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\end{aligned}
$$

Contour Integration (see Fourier Transform arising):

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \frac{\log p}{p^\sigma}\int \phi(s)e^{-it\log p}ds.
$$

Knowledge of zeros gives info on coefficients.

Explicit Formula: Examples

Cuspidal Newforms: Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) = \widehat{\phi}(0)+\frac{1}{2}\phi(0)-\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}P(f;\phi) + O\left(\frac{\log\log R}{\log R}\right)$$

$$P(f;\phi) = \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.$$

Questions

Testing Random Matrix Theory Predictions

Know the right model for large conductors, searching for the correct model for finite conductors.

In the limit must recover the independent model, and want to explain data on:

1. **Excess Rank:** Rank r one-parameter family over $\mathbb{Q}(T)$: observed percentages with rank $\geq r+2$.

2. **First (Normalized) Zero above Central Point:** Influence of zeros at the central point on the distribution of zeros near the central point.

Results and Data

RMT: Theoretical Results ($N \to \infty$, Mean $\to 0.321$)

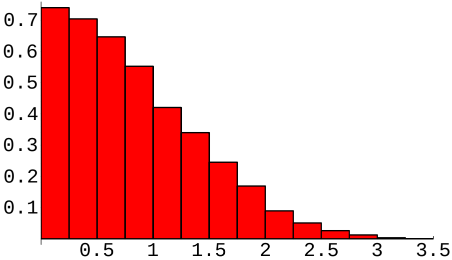


Figure 1a: 1st norm. evalue above 1: 23,040 SO(4) matrices
Mean $=$ .709, Std Dev of the Mean $=$ .601, Median $=$ .709

RMT: Theoretical Results ($N \to \infty$, Mean $\to 0.321$)

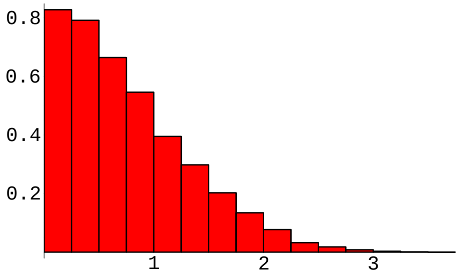


Figure 1b: 1st norm. evalue above 1: 23,040 SO(6) matrices
Mean $=$ .635, Std Dev of the Mean $=$ .574, Median $=$ .635

RMT: Theoretical Results ($N \to \infty$)

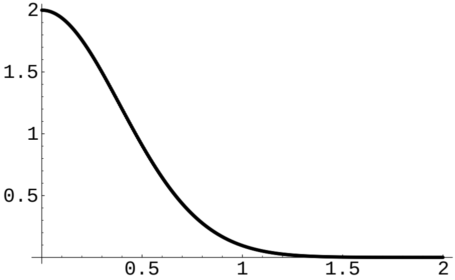


Figure 1c: 1st norm. evalue above 1: SO(even)

RMT: Theoretical Results ($N \to \infty$)

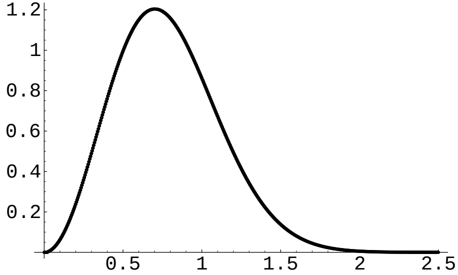


Figure 1d: 1st norm. evalue above 1: SO(odd)

Rank 0 Curves: 1st Normalized Zero above Central Point

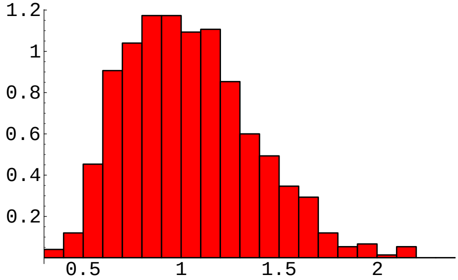


Figure 2a: 750 rank 0 curves from
$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [3.2, 12.6]$, median $= 1.00$ mean $= 1.04$, $\sigma_\mu = .32$

Rank 0 Curves: 1st Normalized Zero above Central Point

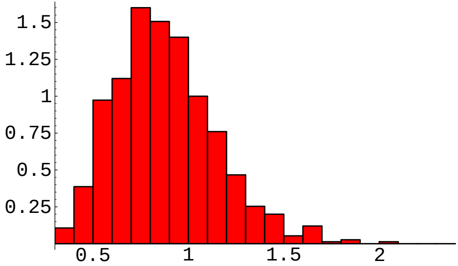


Figure 2b: 750 rank 0 curves from
$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\mathrm{cond}) \in [12.6, 14.9]$, median $= .85$, mean $= .88$, $\sigma_\mu = .27$

Rank 2 Curves: 1st Norm. Zero above the Central Point

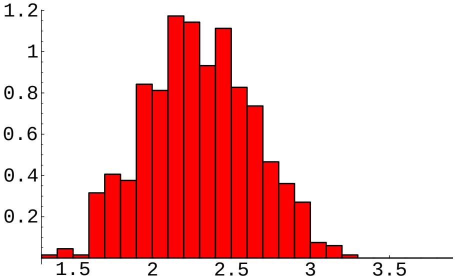


Figure 3a: 665 rank 2 curves from
$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [10, 10.3125]$, median $= 2.29$, mean $= 2.30$

Rank 2 Curves: 1st Norm. Zero above the Central Point

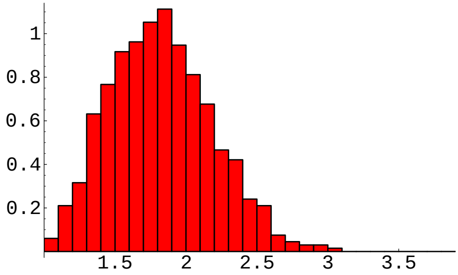


Figure 3b: 665 rank 2 curves from
$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [16, 16.5]$, median $= 1.81$, mean $= 1.82$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

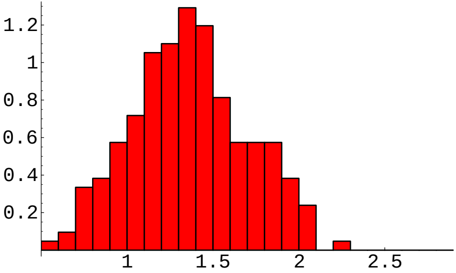


Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\mathrm{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

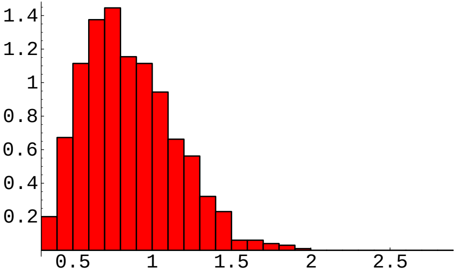


Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

Family	Median $\widetilde{\mu}$	Mean μ	StDev σ_μ	log(cond)	Number
1: [0,1,1,1,T]	1.28	1.33	0.26	[3.93, 9.66]	7
2: [1,0,0,1,T]	1.39	1.40	0.29	[4.66, 9.94]	11
3: [1,0,0,2,T]	1.40	1.41	0.33	[5.37, 9.97]	11
4: [1,0,0,-1,T]	1.50	1.42	0.37	[4.70, 9.98]	20
5: [1,0,0,-2,T]	1.40	1.48	0.32	[4.95, 9.85]	11
6: [1,0,0,T,0]	1.35	1.37	0.30	[4.74, 9.97]	44
7: [1,0,1,-2,T]	1.25	1.34	0.42	[4.04, 9.46]	10
8: [1,0,2,1,T]	1.40	1.41	0.33	[5.37, 9.97]	11
9: [1,0,-1,1,T]	1.39	1.32	0.25	[7.45, 9.96]	9
10: [1,0,-2,1,T]	1.34	1.34	0.42	[3.26, 9.56]	9
11: [1,1,-2,1,T]	1.21	1.19	0.41	[5.73, 9.92]	6
12: [1,1,-3,1,T]	1.32	1.32	0.32	[5.04, 9.98]	11
13: [1,-2,0,T,0]	1.31	1.29	0.37	[4.73, 9.91]	39
14: [-1,1,-3,1,T]	1.45	1.45	0.31	[5.76, 9.92]	10
All Curves	1.35	1.36	0.33	[3.26, 9.98]	209
Distinct Curves	1.35	1.36	0.33	[3.26, 9.98]	196

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

Family	Median $\widetilde{\mu}$	Mean μ	StDev σ_μ	log(cond)	Number
1: [0,1,1,1,T]	0.80	0.86	0.23	[15.02, 15.97]	49
2: [1,0,0,1,T]	0.91	0.93	0.29	[15.00, 15.99]	58
3: [1,0,0,2,T]	0.90	0.94	0.30	[15.00, 16.00]	55
4: [1,0,0,-1,T]	0.80	0.90	0.29	[15.02, 16.00]	59
5: [1,0,0,-2,T]	0.75	0.77	0.25	[15.04, 15.98]	53
6: [1,0,0,T,0]	0.75	0.82	0.27	[15.00, 16.00]	130
7: [1,0,1,-2,T]	0.84	0.84	0.25	[15.04, 15.99]	63
8: [1,0,2,1,T]	0.90	0.94	0.30	[15.00, 16.00]	55
9: [1,0,-1,1,T]	0.86	0.89	0.27	[15.02, 15.98]	57
10: [1,0,-2,1,T]	0.86	0.91	0.30	[15.03, 15.97]	59
11: [1,1,-2,1,T]	0.73	0.79	0.27	[15.00, 16.00]	124
12: [1,1,-3,1,T]	0.98	0.99	0.36	[15.01, 16.00]	66
13: [1,-2,0,T,0]	0.72	0.76	0.27	[15.00, 16.00]	120
14: [-1,1,-3,1,T]	0.90	0.91	0.24	[15.00, 15.99]	48
All Curves	0.81	0.86	0.29	[15.00,16.00]	996
Distinct Curves	0.81	0.86	0.28	[15.00,16.00]	863

Rank 2 Curves: 1st Norm Zero: one-param of rank 0 over $\mathbb{Q}(T)$

first set $\log(\text{cond}) \in [15, 15.5)$; second set $\log(\text{cond}) \in [15.5, 16]$.
Median $\widetilde{\mu}$, Mean μ, Std Dev (of Mean) σ_μ.

Family	$\widetilde{\mu}$	μ	σ_μ	Number	$\widetilde{\mu}$	μ	σ_μ	Number
1: [0,1,3,1,T]	1.59	1.83	0.49	8	1.71	1.81	0.40	19
2: [1,0,0,1,T]	1.84	1.99	0.44	11	1.81	1.83	0.43	14
3: [1,0,0,2,T]	2.05	2.03	0.26	16	2.08	1.94	0.48	19
4: [1,0,0,-1,T]	2.02	1.98	0.47	13	1.87	1.94	0.32	10
5: [1,0,0,T,0]	2.05	2.02	0.31	23	1.85	1.99	0.46	23
6: [1,0,1,1,T]	1.74	1.85	0.37	15	1.69	1.77	0.38	23
7: [1,0,1,2,T]	1.92	1.95	0.37	16	1.82	1.81	0.33	14
8: [1,0,1,-1,T]	1.86	1.88	0.34	15	1.79	1.87	0.39	22
9: [1,0,1,-2,T]	1.74	1.74	0.43	14	1.82	1.90	0.40	14
10: [1,0,-1,1,T]	2.00	2.00	0.32	22	1.81	1.94	0.42	18
11: [1,0,-2,1,T]	1.97	1.99	0.39	14	2.17	2.14	0.40	18
12: [1,0,-3,1,T]	1.86	1.88	0.34	15	1.79	1.87	0.39	22
13: [1,1,0,T,0]	1.89	1.88	0.31	20	1.82	1.88	0.39	26
14: [1,1,1,1,T]	2.31	2.21	0.41	16	1.75	1.86	0.44	15
15: [1,1,-1,1,T]	2.02	2.01	0.30	11	1.87	1.91	0.32	19
16: [1,1,-2,1,T]	1.95	1.91	0.33	26	1.98	1.97	0.26	18
17: [1,1,-3,1,T]	1.79	1.78	0.25	13	2.00	2.06	0.44	16
18: [1,-2,0,T,0]	1.97	2.05	0.33	24	1.91	1.92	0.44	24
19: [-1,1,0,1,T]	2.11	2.12	0.40	21	1.71	1.88	0.43	17
20: [-1,1,-2,1,T]	1.86	1.92	0.28	23	1.95	1.90	0.36	18
21: [-1,1,-3,1,T]	2.07	2.12	0.57	14	1.81	1.81	0.41	19
All Curves	1.95	1.97	0.37	350	1.85	1.90	0.40	388
Distinct Curves	1.95	1.97	0.37	335	1.85	1.91	0.40	366

Rank 2 Curves: 1st Norm Zero: 21 One-Param of Rank 0 over $\mathbb{Q}(T)$

- Observe the medians and means of the small conductor set to be larger than those from the large conductor set.
- For all curves the Pooled and Unpooled Two-Sample t-Procedure give t-statistics of 2.5 with over 600 degrees of freedom.
- For distinct curves the t-statistics is 2.16 (respectively 2.17) with over 600 degrees of freedom (about a 3% chance).
- Provides evidence against the null hypothesis (that the means are equal) at the .05 confidence level (though not at the .01 confidence level).

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$) 1st Normalized Zero above Central Point

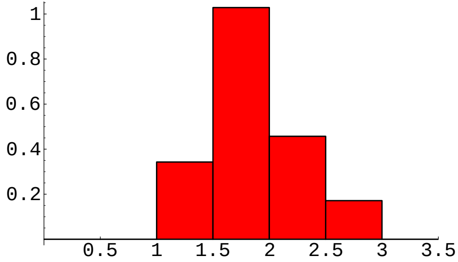


Figure 5*a*: 35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$)
1st Normalized Zero above Central Point

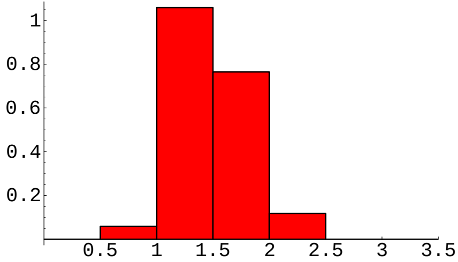


Figure 5b: 34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$

Rank 2 Curves: 1st Norm Zero: rank 2 one-param over $\mathbb{Q}(T)$

$\log(\text{cond}) \in [15, 16]$, $t \in [0, 120]$, median is 1.64.

Family	Mean	Standard Deviation	log(conductor)	Number
1: [1,T,0,-3-2T,1]	1.91	0.25	[15.74,16.00]	2
2: [1,T,-19,-T-1,0]	1.57	0.36	[15.17,15.63]	4
3: [1,T,2,-T-1,0]	1.29		[15.47, 15.47]	1
4: [1,T,-16,-T-1,0]	1.75	0.19	[15.07,15.86]	4
5: [1,T,13,-T-1,0]	1.53	0.25	[15.08,15.91]	3
6: [1,T,-14,-T-1,0]	1.69	0.32	[15.06,15.22]	3
7: [1,T,10,-T-1,0]	1.62	0.28	[15.70,15.89]	3
8: [0,T,11,-T-1,0]	1.98		[15.87,15.87]	1
9: [1,T,-11,-T-1,0]				
10: [0,T,7,-T-1,0]	1.54	0.17	[15.08,15.90]	7
11: [1,T,-8,-T-1,0]	1.58	0.18	[15.23,25.95]	6
12: [1,T,19,-T-1,0]				
13: [0,T,3,-T-1,0]	1.96	0.25	[15.23, 15.66]	3
14: [0,T,19,-T-1,0]				
15: [1,T,17,-T-1,0]	1.64	0.23	[15.09, 15.98]	4
16: [0,T,9,-T-1,0]	1.59	0.29	[15.01, 15.85]	5
17: [0,T,1,-T-1,0]	1.51		[15.99, 15.99]	1
18: [1,T,-7,-T-1,0]	1.45	0.23	[15.14, 15.43]	4
19: [1,T,8,-T-1,0]	1.53	0.24	[15.02, 15.89]	10
20: [1,T,-2,-T-1,0]	1.60		[15.98, 15.98]	1
21: [0,T,13,-T-1,0]	1.67	0.01	[15.01, 15.92]	2
All Curves	1.61	0.25	[15.01, 16.00]	64

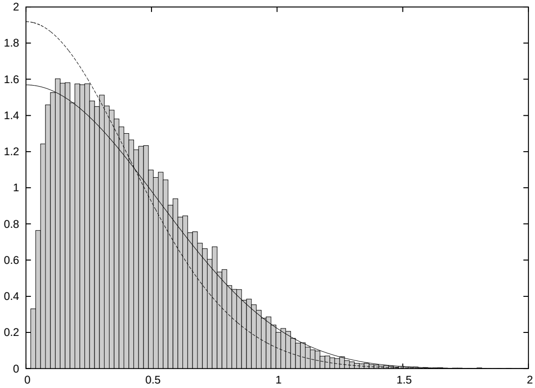
Jacobi Ensembles

New ingredients

- $N_{\text{effective}}$: Use lower order terms in 1-level density to find a better matrix size.

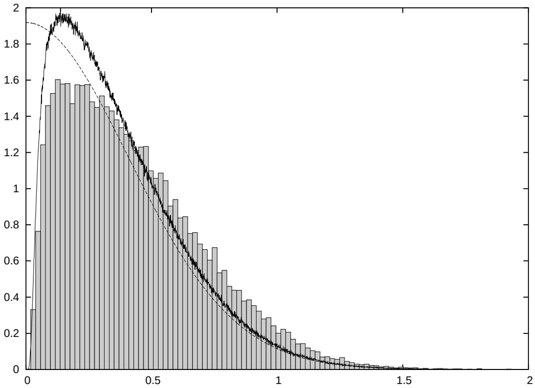
- Discretization of Jacobi Ensemble: Values of L-functions discretized, only consider characteristic polynomials whose absolute value exceeds given quantity.

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



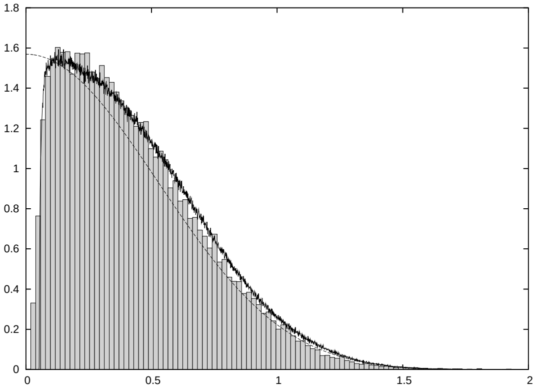
Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with N_{eff} (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with $N_0 = 12$ (solid) with discretisation and with standard $N_0 = 12.26$ (dashed) without discretisation.

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) effective N of N_{eff} = 2 (solid) with discretisation and with effective N of $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

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Caveat: this bibliography hasn't been updated much from a previous talk, and could be a little out of date. It is meant to serve as a first reference.

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