Winning Strategy of Multiplayer and Multialliance Geometric Game– A Game Stemming From the Fibonacci Zeckendorf Game

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The Zeckendorf Game

This game is introduced in "The Zeckendorf Game" paper[1]
Rules: At the beginning of the game, there is an unordered list of n 1's.
Let $F_1 = 1$, $F_2 = 2$, and $F_{i+1} = F_i + F_{i-1}$; therefore the initial list is $\{F_1^n\}$.
On each turn, a player can do one of the following moves:

1. $F_{i-1} \wedge F_i \rightarrow F_{i+1}$
2. If the list has two of the same Fibonacci number, $F_i \wedge F_i$ then
   a. if $i = 1$, $F_1 \wedge F_1 \rightarrow F_2$
   b. if $i = 2$, $F_2 \wedge F_2 \rightarrow F_1 \wedge F_3$
   c. if $i \geq 3$, $F_i \wedge F_i \rightarrow F_{i-2} \wedge F_{i+1}$

The game terminates at the Zeckendorf decomposition(no more moves left).

Zeckendorf Games Always End

Theorem (Baird-Smith, P., Epstein, A., Flint, K., & Miller, S. J. (2018, May). *"The Zeckendorf Game"*.[1])

All games end in finitely many moves.

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive terms: $\left(\sqrt{k}+\sqrt{k}\right)-\sqrt{k+2}<0$.
- Splitting: $2\sqrt{k}-\left(\sqrt{k+1}+\sqrt{k+1}\right)<0$.
- Adding 1's: $2\sqrt{1}-\sqrt{2}<0$.
- Splitting 2's: $2\sqrt{2}-\left(\sqrt{3}+\sqrt{1}\right)<0$.

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Theorem (Baird-Smith, P., Epstein, A., Flint, K., & Miller, S. J. (2018, May). *"The Zeckendorf Game"*.[1])

For all $n > 2$, Player 2 has the winning strategy for 2 player Zeckendorf Game.

- **Idea:** If not, P2 could steal P1's Winning strategy.

Winning strategies of Multiplayer Zeckendorf Games

Theorem

For all $n \geq 5$, $p \geq 3$ Multi-player Game, no player has winning strategy

- **Idea:** Suppose player m has the winning strategy $(1 \leq m \leq p)$. Then player m-1 can steal player m's winning strategy
  1. Since for all $n \geq 5$, $p \geq 3$ games, any player m's winning path does not contain the following 3 consecutive steps(unless player m is the player who takes step 2). If it contains, player in step 2 can do $F_1 \wedge F_2 \rightarrow F3$ instead and player $m-1$ can steal the winning strategy:
     Step $1: F_1 \wedge F_1 \rightarrow F_2$ (Combine two 1s into one 2)
     Step $2: F_1 \wedge F_1 \rightarrow F_2$ (Combine two 1s into one 2)
     Step $3: F_2 \wedge F_2 \rightarrow F_1 \wedge F_3$ (Split two 2s into one 1 and one 3)
  2. Then we construct other $m-1$ players' moves containing these 3 consecutive steps, which contradicts above, so player m has no winning strategy

Winning Strategies of Multialliance Zeckendorf Games

Theorem

In a game consisting of t teams and exactly k consecutive players each team. When n is significantly large, for any $t \geq 3, k = t - 1$, no team has winning strategy

- **Idea:** Suppose team m has the winning strategy $(1 \leq m \leq t)$. Then team m-1 can steal team m's winning strategy
  1. Since for any $t \geq 3, k = t - 1$, any team m's winning path doesn't contain the following $3k$ consecutive steps (unless one of the middle k players is in team m). If it contains, the middle k players listed below can all do $F_1 \wedge F_2 \rightarrow F3$ instead and team $m - 1$ can steal the winning strategy:
     First k steps all do : $F_1 \wedge F_1 \rightarrow F_2$ (Combine two 1s into one 2)
     Middle k steps all do : $F_1 \wedge F_1 \rightarrow F_2$ (Combine two 1s into one 2)
     Last k steps all do : $F_2 \wedge F_2 \rightarrow F_1 \wedge F_3$ (Split two 2s into 1 and 3)
  2. Then we construct these $3k$ steps for other $m - 1$ teams and we get contradiction

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Geometric Game with Common Ratio 2

Geometric Game Rules: At the beginning of the game, there is an unordered list of n 1's. Let $a_1 = 1$, $a_2 = 2$, and $a_{i+1} = 2a_i$; therefore the initial list is $\{a_1^n\}$.
On each turn, a player can do one of the following moves:

1. Combining moves: if the list has two copies of the same Geometric Number, $a_i \wedge a_i \rightarrow a_{i+1}$
2. Splitting moves: if the list has three copies of the same Geometric number that is greater than 1, then $a_i \wedge a_i \wedge a_i \rightarrow a_{i+1} \wedge 2a_{i-1}$

The game terminates at the Geometric decomposition(no more moves left). Players take turns playing the game and the player who does the last move wins the game.

Existence and Uniqueness of the Geometric Game

Theorem

For each starting number n, there exists a unique Geometric Decomposition.

Proof Idea: First prove existence, then prove uniqueness, both using strong inductions.

Geometric Games Always End

Theorem

All geometric games end in finitely many moves.

Proof Idea: The number of terms and the sum of indices are both decreasing monovariants.

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Winning strategies of Multiplayer Geometric Games

Theorem

For all $n \geq 28$, $p \geq 5$ Multiplayer Geometric Game, no player has winning strategy

- **Idea:** Suppose player m has the winning strategy $(1 \leq m \leq p)$. Then player m-1 can steal player m's winning strategy
  1. Since for all $n \geq 28$, $p \geq 6$ games, if one player has a winning strategy, the rest of the players can do the following 5 consecutive steps.
     Step $1: 1 \wedge 1 \to 2$
     Step $2: 1 \wedge 1 \to 2$
     Step $3: 1 \wedge 1 \to 2$
     Step $4: 2 \wedge 2 \wedge 2 \to 1 \wedge 1 \wedge 4$
     Step $5: 1 \wedge 1 \to 2$
  2. Then the player in step 4 can do $2 \wedge 2 \to 4$ instead.
  3. Then prove that when $n \geq 28$, $p = 5$, no player has a winning strategy.

More results on Winning strategies of Multiplayer Geometric Games

Theorem

When $n \geq 8$ and $p = 3$, player 2 never has a winning strategy.

Theorem

When $n \geq 10$ and $p = 4$, player 1 and player 2 never has a winning strategy.

Proof Idea: first suppose that player 1 has a winning strategy and prove by contradiction; then suppose that player 2 has a winning strategy and prove by contradiction.

Winning strategies of Multialliance Geometric Games

Theorem

When there are t alliances and each alliance has $k = t - 1$ consecutive players, if $t \geq 5$ and $n \geq max\{2k^2 + 10k, 160\}$, then no alliance has a winning strategy.

Winning strategies of Multialliance Geometric Games

Theorem

When $n \geq 12k^2 + 13k$ and there are 2 alliances in the Geometric Game, if one alliance consists of $3k+1$ consecutive players (call it big alliance), and the other alliance has k consecutive players, then the big alliance always has a winning strategy.

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Future Directions

Future Direction

1. Winning strategy for the 2-player Geometric Game
2. Construction of alliances with winning strategy in multiplayer game ($p > 2$).
3. More results for the winning strategy of 3-player and 4-player Geometric Game.
4. Further tighten the bounds of n and p for the result of winning strategies.

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