# Benford's Law, Values of $L$-Functions and the $3x + 1$ Problem

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## Summary

- Acknowledgements: This talk is based on joint work with Alex Kontorovich

- Outline:
  - ⋄ Review history of Benford's Law.
  - ⋄ Discuss applications of Poisson Summation.
  - ⋄ Study $L$-functions and $3x + 1$.

## Benford's Law: Newcomb (1881), Benford (1938)

### Statement

For many data sets, probability of observing a first digit of $d$ base $B$ is $\log_B\left(\frac{d+1}{d}\right)$; base 10 about 30% are 1s.

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- Not all data sets satisfy Benford's Law.
  - ◇ Long street $[1, L]$: $L = 199$ versus $L = 999$.
  - ◇ Oscillates between $1/9$ and $5/9$ with first digit 1.
  - ◇ Many streets of different sizes: close to Benford.

## Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- $L$-functions, characteristic polynomials
- iterates of the $3x + 1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

## Applications

- analyzing round-off errors
- determining the optimal way to store numbers
- detecting tax and image fraud, and data integrity

# General Theory

## Mantissas

For $x > 0$ write $x = M_B(x) \cdot B^k$, $M_B(x) \in [1, B)$ is the mantissa and $k \in \mathbb{Z}$.

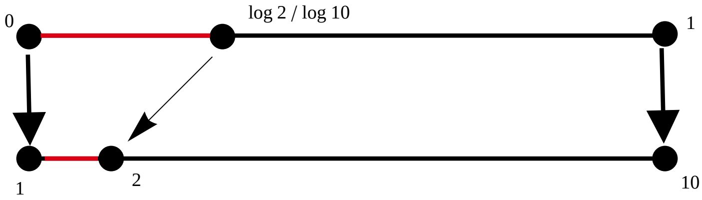
$M_B(x) = M_B(\widetilde{x})$ if and only if $x$ and $\widetilde{x}$ have the same leading digits.

Key observation: $\log_B(x) = \log_B(\widetilde{x}) \bmod 1$ if and only if $x$ and $\widetilde{x}$ have the same leading digits. Thus often study $y = \log_B x$.

## Logarithms and Benford's Law

### Fundamental Equivalence

Data set $\{x_i\}$ is Benford base $B$ if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.



# Benford Good Processes

## Poisson Summation and Benford's Law: Definitions

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- Poisson Summation Formula: $f$ nice:

$$\sum_{\ell=-\infty}^{\infty} f(\ell) \ = \ \sum_{\ell=-\infty}^{\infty} \widehat{f}(\ell),$$

Fourier transform $\widehat{f}(\xi) \ = \ \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$

## Benford Good Process

$X_T$ is Benford Good if there is a nice $f$ st

$$\mathrm{CDF}_{\overrightarrow{Y}_{T,B}}(y) = \int_{-\infty}^{y} \frac{1}{T} f\left(\frac{t}{T}\right) dt + E_T(y) := G_T(y)$$

and monotonically increasing $h$ ($h(|T|) \to \infty$):

- Small tails: $G_T(\infty) - G_T(Th(T)) = o(1)$, $G_T(-Th(T)) - G_T(-\infty) = o(1)$.
- Decay of the Fourier Transform: $\sum_{\ell \neq 0} \left| \frac{\widehat{f}(T\ell)}{\ell} \right| = o(1)$.
- Small translated error: $\mathcal{E}(a, b, T)) = \sum_{|\ell| \leq Th(T)} [E_T(b + \ell) - E_T(a + \ell)] = o(1)$.

## Main Theorem

**Theorem (Kontorovich and M–, 2005)**

*$X_T$ converging to $X$ as $T \to \infty$ (think spreading Gaussian). If $X_T$ is Benford good, then $X$ is Benford.*

- Examples
  - ◇ $L$-functions
  - ◇ characteristic polynomials (RMT)
  - ◇ $3x+1$ problem
  - ◇ geometric Brownian motion.

## Sketch of the proof

- Structure Theorem:
  - ◇ main term is something nice spreading out
  - ◇ apply Poisson summation
- Control translated errors:
  - ◇ hardest step
  - ◇ techniques problem specific

## Sketch of the proof (continued)

$$\sum_{\ell=-\infty}^{\infty} \mathbb{P}\left(a+\ell \le \overrightarrow{Y}_{T,B} \le b+\ell\right)$$

$$= \sum_{|\ell| \le Th(T)} \left[G_T(b+\ell) - G_T(a+\ell)\right] + o(1)$$

$$= \int_a^b \sum_{|\ell| \le Th(T)} \frac{1}{T} f\left(\frac{t}{T}\right) dt + \mathcal{E}(a,b,T) + o(1)$$

$$= \widehat{f}(0)\cdot(b-a) + \sum_{\ell \ne 0} \widehat{f}(T\ell)\frac{e^{2\pi i b\ell} - e^{2\pi i a\ell}}{2\pi i\ell} + o(1).$$

# Values of $L$-Functions

## 'Good' $L$-Functions

We say an $L$-function is *good* if:

- Euler product:

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_p \prod_{j=1}^{d} \left(1 - \alpha_{f,j}(p)p^{-s}\right)^{-1}.$$

- $L(s,f)$ has a meromorphic continuation to $\mathbb{C}$, is of finite order, and has at most finitely many poles (all on the line $\mathrm{Re}(s) = 1$).
- Functional equation:

$$e^{\mathrm{i}\omega} G(s) L(s,f) \ = \ e^{-\mathrm{i}\omega} \overline{G(1-\overline{s}) L(1-\overline{s})},$$

where $\omega \in \mathbb{R}$ and

$$G(s) \ = \ Q^s \prod_{i=1}^{h} \Gamma(\lambda_i s + \mu_i)$$

with $Q, \lambda_i > 0$ and $\mathrm{Re}(\mu_i) \geq 0$.

## 'Good' $L$-Functions (cont)

- For some $\aleph > 0$, $c \in \mathbb{C}$, $x \geq 2$ we have

$$\sum_{p \leq x} \frac{|a_f(p)|^2}{p} \ = \ \aleph \log\log x + c + O\left(\frac{1}{\log x}\right).$$

- The $\alpha_{f,j}(p)$ are (Ramanujan-Petersson) tempered: $|\alpha_{f,j}(p)| \leq 1$.
- If $N(\sigma, T)$ is the number of zeros $\rho$ of $L(s)$ with $\mathrm{Re}(\rho) \geq \sigma$ and $\mathrm{Im}(\rho) \in [0, T]$, then for some $\beta > 0$ we have

$$N(\sigma, T) \ = \ O\left(T^{1-\beta\left(\sigma - \frac{1}{2}\right)} \log T\right).$$

Known in some cases, such as $\zeta(s)$ and Hecke cuspidal forms of full level and even weight $k > 0$.

## Results

**Theorem (Kontorovich-Miller '05)**

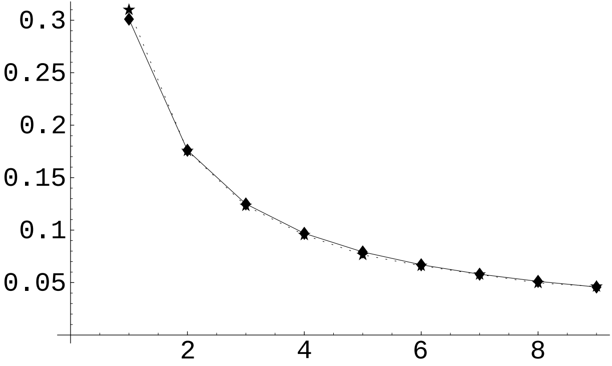
Let $L(s, f)$ be a good $L$-function. Fix a $\delta \in (0, 1)$. For each $T$, let $\sigma_T = \frac{1}{2} + \frac{1}{\log^\delta T}$. Then as $T \to \infty$

$$\frac{\mu\left\{t \in [T, 2T] : \; M_B\left(|L(\sigma_T + \mathrm{i}t, f)|\right) \le \tau\right\}}{T} \to \log_B \tau$$

Thus the values of the $L$-function satisfy Benford's Law in the limit for any base $B$.

## Riemann Zeta Function: Data

$$\left|\zeta\left(\frac{1}{2} + i\frac{k}{4}\right)\right|, k \in \{0, 1, \ldots, 65535\}.$$



# The $3x + 1$ Problem

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- $7 \to_1 11 \to_1 17 \to_2 13 \to_3 5 \to_4 1 \to_2 1$,
  2-path $(1, 1)$, 5-path $(1, 1, 2, 3, 4)$.
  $m$-path: $(k_1, \dots, k_m)$.

# Heuristic Proof of $3x+1$ Conjecture

$$\begin{aligned} a_{n+1} &= T(a_n) \\ \mathbb{E}[\log a_{n+1}] &\approx \sum_{k=1}^{\infty} \frac{1}{2^k} \log\left(\frac{3a_n}{2^k}\right) \\ &= \log a_n + \log 3 - \log 2 \sum_{k=1}^{\infty} \frac{k}{2^k} \\ &= \log a_n + \log\left(\frac{3}{4}\right). \end{aligned}$$

Geometric Brownian Motion, drift $\log(3/4) < 1$.

## Structure Theorem: Sinai, Kontorovich-Sinai

$\mathbb{P}(A) = \lim_{N\to\infty} \frac{\#\{n\le N: n\equiv 1,5 \bmod 6, n\in A\}}{\#\{n\le N: n\equiv 1,5 \bmod 6\}}$.

$(k_1, \dots, k_m)$: two full arithm progressions:

$6 \cdot 2^{k_1+\cdots+k_m} p + q$.

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## $3x + 1$ and Benford

**Theorem (Kontorovich and M–, 2005)**

*As $m \to \infty$, $x_m/(3/4)^m x_0$ is Benford.*

**Theorem (Lagarias-Soundararajan 2006)**

*$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first $N$ iterates of the $3x+1$ map are within $2N^{-1/36}$ of the Benford probabilities.*

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- Quantified Equidistribution:

$I_\ell = \{\ell M, \dots, (\ell+1)M - 1\}$, $M = m^c$, $c < 1/2$

$k_1, k_2 \in I_\ell$: $\left|\eta\left(\frac{k_1}{\sqrt{m}}\right) - \eta\left(\frac{k_2}{\sqrt{m}}\right)\right|$ small

$C = \log_B 2$ of irrationality type $\kappa < \infty$:

$\#\{k \in I_\ell : \overline{kC} \in [a,b]\} = M(b-a) + O(M^{1+\epsilon-1/\kappa})$.

## Irrationality Type

### Irrationality type

$\alpha$ has irrationality type $\kappa$ if $\kappa$ is the supremum of all $\gamma$ with

$$\underline{\lim}_{q \to \infty} q^{\gamma+1} \min_p \left| \alpha - \frac{p}{q} \right| = 0.$$

- Algebraic irrationals: type 1 (Roth's Thm).
- Theory of Linear Forms: $\log_B 2$ of finite type.

## Linear Forms

### Theorem (Baker)

*$\alpha_1, \dots, \alpha_n$ algebraic numbers height $A_j \geq 4$, $\beta_1, \dots, \beta_n \in \mathbb{Q}$ with height at most $B \geq 4$,*

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n.$$

*If $\Lambda \neq 0$ then $|\Lambda| > B^{-C\Omega \log \Omega'}$, with $d = [\mathbb{Q}(\alpha_i, \beta_j) : \mathbb{Q}]$, $C = (16nd)^{200n}$, $\Omega = \prod_j \log A_j$, $\Omega' = \Omega / \log A_n$.*

Gives $\log_{10} 2$ of finite type, with $\kappa < 1.2 \cdot 10^{602}$:

$$|\log_{10} 2 - p/q| = |q \log 2 - p \log 10| \,/ q \log 10.$$

## Quantified Equidistribution

### Theorem (Erdös-Turan)

$$D_N = \frac{\sup_{[a,b]} |N(b-a) - \#\{n \le N : x_n \in [a,b]\}|}{N}$$

*There is a C such that for all m:*

$$D_N \le C \cdot \left( \frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right)$$

## Proof of Erdös-Turan

Consider special case $x_n = n\alpha$, $\alpha \notin \mathbb{Q}$.

- Exponential sum $\le \frac{1}{|\sin(\pi h\alpha)|} \le \frac{1}{2||h\alpha||}$.
- Must control $\sum_{h=1}^{m} \frac{1}{h||h\alpha||}$, see irrationality type enter.
- type $\kappa$, $\sum_{h=1}^{m} \frac{1}{h||h\alpha||} = O\left(m^{\kappa-1+\epsilon}\right)$, take $m = \lfloor N^{1/\kappa} \rfloor$.

## $3x+1$ Data: random 10,000 digit number, $2^k || 3x+1$

80,514 iterations ($(4/3)^n = a_0$ predicts 80,319); $\chi^2 = 13.5$ (5% 15.5).

| Digit | Number | Observed | Benford |
|---|---|---|---|
| 1 | 24251 | 0.301 | 0.301 |
| 2 | 14156 | 0.176 | 0.176 |
| 3 | 10227 | 0.127 | 0.125 |
| 4 | 7931 | 0.099 | 0.097 |
| 5 | 6359 | 0.079 | 0.079 |
| 6 | 5372 | 0.067 | 0.067 |
| 7 | 4476 | 0.056 | 0.058 |
| 8 | 4092 | 0.051 | 0.051 |
| 9 | 3650 | 0.045 | 0.046 |

## $3x+1$ Data: random 10,000 digit number, $2|3x+1$

241,344 iterations, $\chi^2 = 11.4$ (5% 15.5).

| Digit | Number | Observed | Benford |
|---|---|---|---|
| 1 | 72924 | 0.302 | 0.301 |
| 2 | 42357 | 0.176 | 0.176 |
| 3 | 30201 | 0.125 | 0.125 |
| 4 | 23507 | 0.097 | 0.097 |
| 5 | 18928 | 0.078 | 0.079 |
| 6 | 16296 | 0.068 | 0.067 |
| 7 | 13702 | 0.057 | 0.058 |
| 8 | 12356 | 0.051 | 0.051 |
| 9 | 11073 | 0.046 | 0.046 |

## $5x + 1$ Data: random 10,000 digit number, $2^k || 5x + 1$

27,004 iterations, $\chi^2 = 1.8$ (5% 15.5).

| Digit | Number | Observed | Benford |
|---|---|---|---|
| 1 | 8154 | 0.302 | 0.301 |
| 2 | 4770 | 0.177 | 0.176 |
| 3 | 3405 | 0.126 | 0.125 |
| 4 | 2634 | 0.098 | 0.097 |
| 5 | 2105 | 0.078 | 0.079 |
| 6 | 1787 | 0.066 | 0.067 |
| 7 | 1568 | 0.058 | 0.058 |
| 8 | 1357 | 0.050 | 0.051 |
| 9 | 1224 | 0.045 | 0.046 |

## $5x+1$ Data: random 10,000 digit number, $2|5x+1$

241,344 iterations, $\chi^2 = 3 \cdot 10^{-4}$ (5% 15.5).

| Digit | Number | Observed | Benford |
|---|---|---|---|
| 1 | 72652 | 0.301 | 0.301 |
| 2 | 42499 | 0.176 | 0.176 |
| 3 | 30153 | 0.125 | 0.125 |
| 4 | 23388 | 0.097 | 0.097 |
| 5 | 19110 | 0.079 | 0.079 |
| 6 | 16159 | 0.067 | 0.067 |
| 7 | 13995 | 0.058 | 0.058 |
| 8 | 12345 | 0.051 | 0.051 |
| 9 | 11043 | 0.046 | 0.046 |

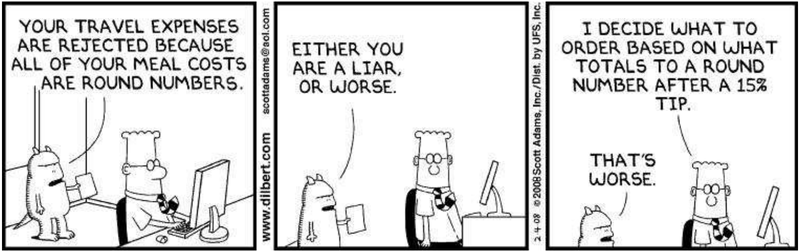
# Conclusions

## Conclusions

- See many different systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).
- Applications to fraud detection / data integrity.

## Caveats!

- Not all fraud can be detected by Benford's Law.
- A math test indicating fraud is *not* proof of fraud: unlikely events, alternate reasons.



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