Centered Moments of Weighted One-Level Densities of GL(2) L-Functions

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Introduction

What is an L-function?

- The Riemann zeta function with Euler product:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

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- Functional Equation:

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- Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

What is a general L-function?

- A general L-function with Euler product

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

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- Functional Equation:

$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f)L(s,f) \;=\; \Lambda(1-s,f).$$

- Grand Riemann Hypothesis (GRH):

All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

Intro to Modular Forms

Definition

A modular form of weight k and level N is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ that is

1. **"periodic"** with respect to the Nth congruence subgroup of $SL_2(\mathbb{Z})$:

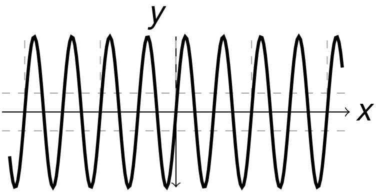
$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

$$\text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), c \equiv 0 \bmod N.$$

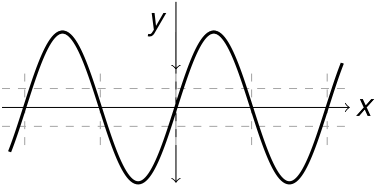
2. **holomorphic** at all cusps: The function f is holomorphic at all cusps of $\Gamma_0(N)$, including ∞.

Oldform-Newform Theory

- Form of level $M \implies$ Form of level ℓM.
- If the form, i.e., has fully reduced level, it is a **newform**; else, **oldform**.



$y = \sin(4\pi x)$



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Twin Analogy

Twins are born; one of them has birthday only once in every 4 years.

Feb 28 11:59 PM **Feb 29 12:00 AM**

Twin Analogy- Oldforms; Counting age using birthday.

After 24 years...

Age 24 Age 6

Twin Analogy - Newforms; Counting age using years

After 24 years...

Age 24 Age 24

Hecke Operator

- A Holomorphic cusp forms have a Fourier expansion of the form
$$f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}.$$

Definition

Define the **n^th Hecke Operator** $T_n : S_k(N) \to S_k(N)$ to be:

$$T_n f(\tau) = \sum_{m=0}^{\infty} \left(\sum_{d|\gcd(m,n)} d^{k-1} a_{mn/d^2} \right) q^m.$$

Hecke eigenvalue

- If f holomorphic cusp newform, for every $n \in \mathbf{N}$, f is eigenfunction of T_n.

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- Hecke eigenvalues are multiplicative:

$$\lambda_f(m)\lambda_f(n) = \sum_{d \mid \gcd(m,n)} \lambda_f\left(\frac{mn}{d^2}\right)$$

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- Hecke eigenvalues are multiplicative:

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- Using Hecke eigenvalues, we can define the L-function:

$$L(s,f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p L_p(s,f)^{-1}$$

Statistics of Zeros

- Assuming GRH, (non-trivial) zeros of L-functions can be written as $\frac{1}{2} + i\gamma_i$.
- We study statistics of γ_i for L-functions associated to modular forms.

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Conjecture (Katz-Sarnak)

(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

Relationship to Random Matrix Theory

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(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

- In 1972, Montgomery and Dyson discovered this phenomenon.

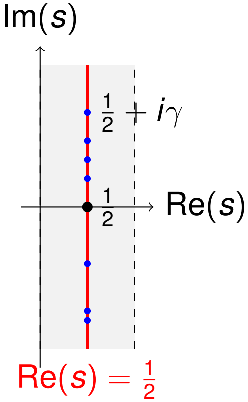
Distribution of Zeros

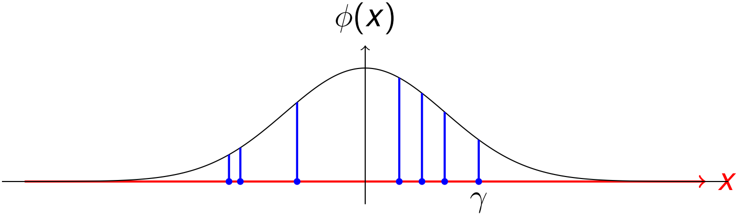
$\longleftrightarrow$ Eigenvalue Distribution of Random Matrix Ensembles

($\longleftrightarrow$ Energy Levels of Heavy Nuclei).

1-level density

- We wished to study the low-lying zeros, near the central point ($s = \frac{1}{2}$).





Definition (1-Level Density)

Let $L_f(s)$ be an L-function associated to a modular form f. Let ϕ be an even Schwartz function with compact Fourier transform. Then, its **1-level density** is:

$$D_1(f;\phi) = \sum_{\gamma_f} \phi\left(\frac{\log R_f}{2\pi}\gamma_f\right).$$

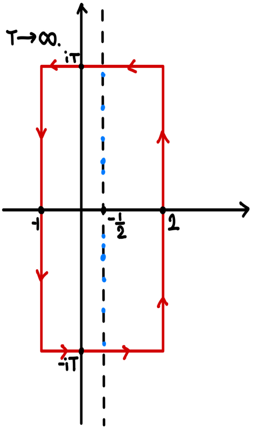
Explicit formula for 1-level density

- Euler product:

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s, f)^{-1}$$

- Integral of the logarithmic derivative against ϕ:

$$\oint \frac{L(s, f)'}{L(s, f)}$$



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  - (LHS) Argument Principle:

$$\sum_{\text{zeros of } L(f,s)} \phi(\cdot)$$

  - (RHS) Products $\to$ Sums:

$$\sum_p \oint \frac{L_p(s,f)'}{L_p(s,f)}$$

Explicit Formula

Theorem (Iwaniec, Luo, and Sarnak [ILS00])

Given the same conditions,

$$D_1(f;\phi) = \frac{A}{\log R} - 2\sum_p \sum_{m=1}^{\infty} \left(\frac{\alpha_f(p)^m + \beta_f(p)^m}{p^{m/2}} \right) \hat{\phi}\left(m\frac{\log p}{\log R} \right) \frac{\log p}{\log R}$$

where A represents a sum of digamma $(\Gamma'(s)/\Gamma(s))$ *factors.*

- We often consider the average of 1-level density: $\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_1(f,\phi)$.

Weights and n^{th} Centered Moment

Introducing Weights

- Weights $\{w_f\}_{\mathcal{F}_N}$ are often used to simplify calculations.

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$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} D_1(f,\phi) = \lim_{N\to\infty} \frac{1}{\left(\sum_{f\in\mathcal{F}_N} 1\right)} \sum_{f\in\mathcal{F}_N} 1 \cdot D_1(f,\phi)$$

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vs.

$$\lim_{N\to\infty} \frac{1}{\left(\sum_{f\in\mathcal{F}_N} w_f\right)} \sum_{f\in\mathcal{F}_N} w_f D_1(f,\phi).$$

Does weight change convergence?

- In 2018, Knightly and Reno asked whether putting in weights can change convergence [KR18].
- Consider two classes with different grading schemes:

	Class U	Class W
Psets	25%	0%
Midterm	25%	0%
Project	25%	0%
Final	25%	100%

Example: Vishal and Say-Yeon

- Suppose we have

	Vishal	Say-Yeon
Psets	100%	0%
Midterm	95%	10%
Project	95%	15%
Final	15%	95%

	Class U	Class W
Psets	25%	0%
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- Who would have gotten a better grade in the class?

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	Class U	Class W
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- Who would have gotten a better grade in the class?
- Depends on the grading scheme!
  In Class U, Vishal would have done better with a 76.25% and Say-Yeon with 30%.
  However, in Class W, Say-Yeon would have done better with 95% and Vishal with 15%.

Weights (Knightly Reno)

- Given a primitive real Dirichlet character χ of modulus $D \geq 1$ and $r > 0$ relatively prime to D. For a holomorphic newform, define the weight

$$w_f = \frac{\Lambda\left(\frac{1}{2}, f \times \chi\right) |a_f(r)|^2}{\| f \|^2}$$

for the completed L-function $\Lambda(s, f \times \chi)$

Theorem ([KR18])

For $\mathcal{F}_n = \mathcal{F}_k(N)^{new}$ ($N + k \to \infty$ as $n \to \infty$), we have

$$\lim_{n\to\infty} \frac{\sum_{f\in\mathcal{F}_n} D_1(f,\phi) w_f}{\sum_{f\in\mathcal{F}_n} w_f} = \begin{cases} \int_{-\infty}^{\infty} \phi(x) W_{\mathrm{Sp}}(x)\, dx, & \text{if } \chi \text{ is trivial,} \\ \int_{-\infty}^{\infty} \phi(x) W_{\mathrm{O}}(x)\, dx, & \text{if } \chi \text{ is nontrivial.} \end{cases}$$

The n^{th} Centered Moment

Definition (n^{th} Centered Moment)

Let $\mathcal{F}_{k,N}$ be the family of holomorphic cusp newforms. Let ϕ be an even Schwartz function with compact Fourier support. Then, its **n^{th} centered moment** is given by:

$$\mathbb{E}_{\mathcal{F}_{k,N}}\left([D_1(f,\phi) - \mathbb{E}_{\mathcal{F}_{k,N}}(D_1(f,\phi))]^n\right)$$

where $\mathbb{E}_{\mathcal{F}_{k,N}}(Q(f)) = \frac{1}{|\mathcal{F}_{k,N}|}\sum_{f\in\mathcal{F}_{k,N}} Q(f)$ for some function $Q : \mathcal{F}_{k,N} \to \mathbb{C}$.

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- Hughes and Miller computes the n^{th} moment density for $\mathcal{F}_{k,N}$ [HM07].

Our Work

- We look at Weighted n^{th} Centered Moments of $\mathcal{F}_{k,N}$.
- We use the same weights with Knightly:

$$w_f = \frac{\Lambda\left(\frac{1}{2}, f \times \chi\right) |a_f(r)|^2}{\| f \|^2}.$$

- We denote

$$\mathcal{E}_w(Q) := \lim_{N \to \infty} \frac{\sum_{f \in \mathcal{F}_{k,N}} Q(f) w_f}{\sum_{f \in \mathcal{F}_{k,N}} w_f}$$

where $Q : \mathcal{F} \to \mathbb{C}$.

Our Work

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Theorem (SMALL 2025)

Let ϕ be a Schwartz test function with $\operatorname{supp}\hat{\phi} \subset (-\frac{1}{2n}, \frac{1}{2n})$. For real Dirichlet character χ, we have

$$\mathcal{E}_w\left[(D(f,\phi) - \mathcal{E}_w(D(f,\phi)))^m\right] = \begin{cases} (2m-1)!!\left(\int_{-\infty}^{\infty} \hat{\phi}^2(y)|y|\,dy\right)^{m/2} & \text{if } m \text{ even,} \\ 0 & \text{if } m \text{ odd} \end{cases}$$

- This confirms the work of [KR18] since symplectic and orthogonal moments agree with the Gaussian on this support.

Our Work

Lemma (SMALL 2025)

For any positive integer $n = \prod_{j=1}^{\ell} q_j^{m_j}$,

$$\mathcal{E}_w\left[\lambda.(n)\right] = \frac{\chi(n)\sigma_1\left(\gcd\left(r,n\right)\right)}{\sqrt{n}} + O\left(\frac{n^{\frac{k-1}{2}}V^k}{N^{\frac{k-1}{2}}k^{\frac{k}{2}-1}}\right),$$

where V *is a constant depending on* r *and* D, *and* σ_1 *is the divisor sum function.*

Proof idea: Apply the multiplicativity of Hecke operators to generalize Prop 3.1 of Knightly and Reno ([KR18]).

Nontrivial Character Analysis

Case	Main Term	Error Term
$m_j + n_j \geq 3$ for some j	0	$\log^{-3} R$
$(m_j, n_j) = (1, 1)$ for some j	0	$\frac{\log\log(3N)}{\log R}$
$(m_j, n_j) = (0, 2)$ for all j	$\begin{cases} 0 & t \text{ odd} \\ (t-1)!!(2\sigma_\phi^2)^{t/2} & t \text{ even} \end{cases}$	$\frac{\log\log(3N)}{\log R}$

Trivial Character Analysis

Case	Main Term	Error Term
$m_j + n_j \geq 3$ for some j	0	$\log^{-3} R$
$m_j + n_j \leq 2$ for all j	$\sum_{s=0}^{\lfloor t/2 \rfloor} \frac{t!}{2^s (t-s)!} \binom{t-s}{s} \left(\frac{\phi(0)}{2}\right)^{t-2s} \left(\frac{\sigma_\phi^2}{2}\right)^s$	$\frac{\log\log(3N)}{\log R}$

Closing

Future work

- Verify Gaussian behavior for Knightly and Reno's other weight

$$w_f = \frac{\Lambda\left(1/2, f \times \chi\right)\Lambda\left(\frac{1}{2}, f\right)}{\parallel f \parallel^2}.$$

- Extend support of test function used from $\left(-\frac{1}{2n}, \frac{1}{2n}\right)$ to $\left(-\frac{1}{n}, \frac{1}{n}\right)$.

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