Signal Recovery Using the Gabor Transform

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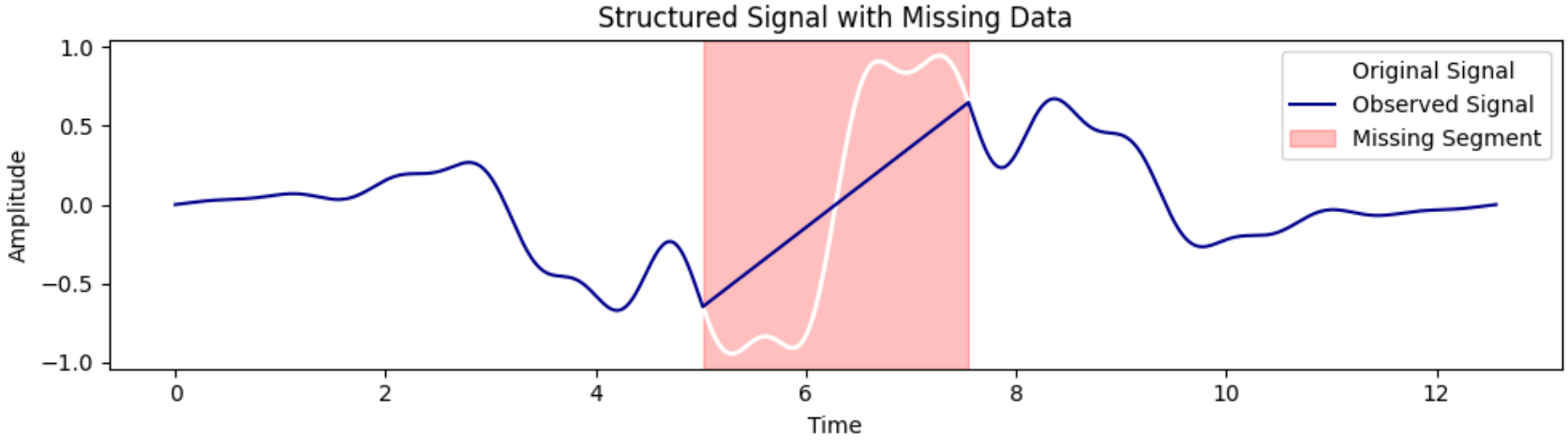
Introduction

Can You Recover the Original Signal?

- You receive only part of a signal - the rest is missing.



(Illustrative example of a corrupted signal)

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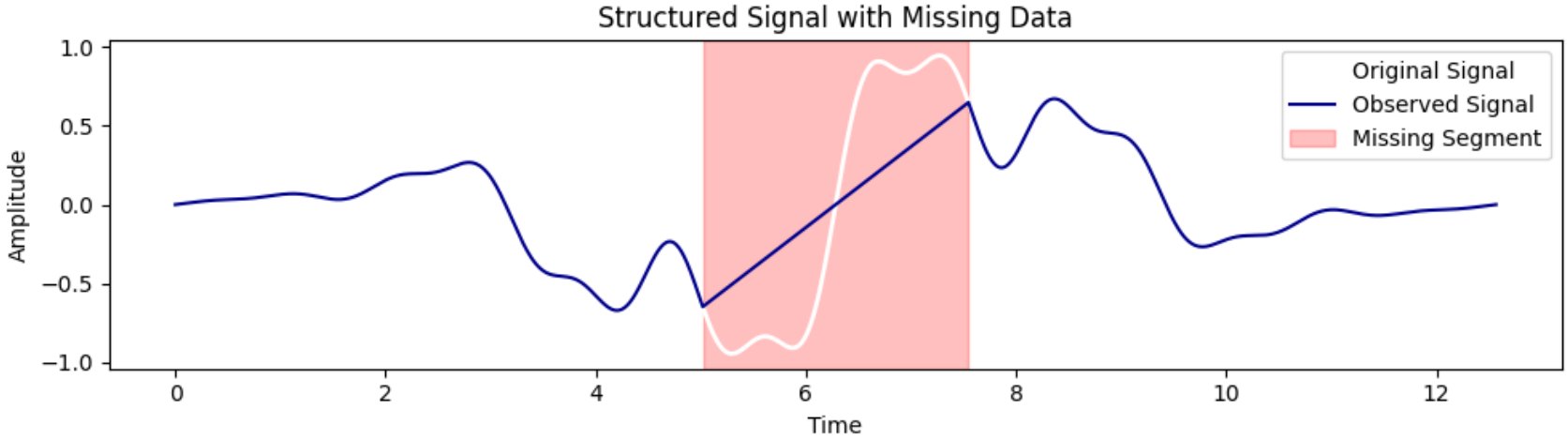


(Illustrative example of a corrupted signal)

- If we know certain properties about a signal and its Fourier transform, we can reconstruct the signal.

Discrete Fourier Transform

- Useful for digital signal processing.
- Can be computed using *fast Fourier transform* algorithms.

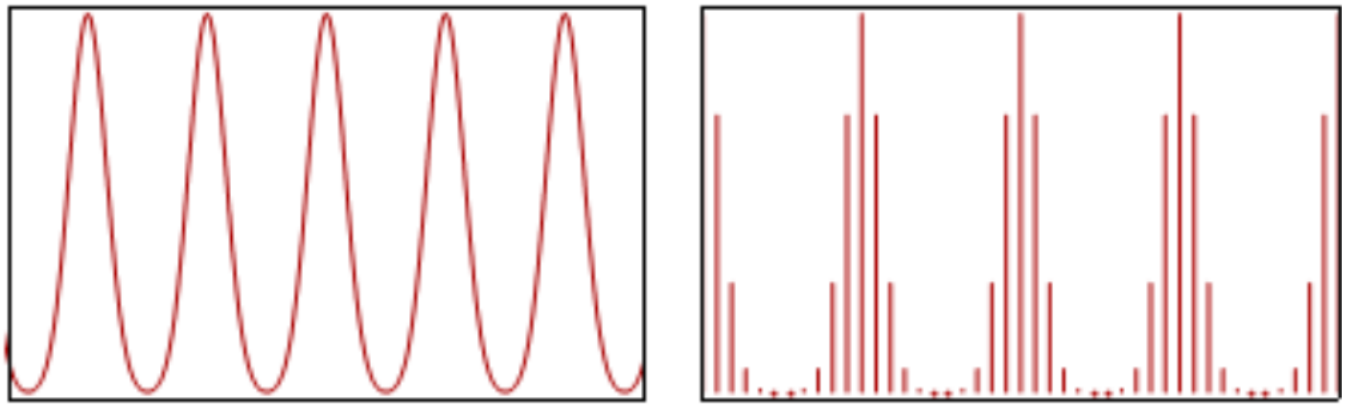


Image source: `https://en.wikipedia.org/wiki/Discrete_Fourier_transform`

Discrete Fourier Transform

Definition (Discrete Fourier Transform)

For a function $f : \mathbb{Z}_N^d \to \mathbb{C}$, the **normalized DFT** is the function $\widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ defined as follows:

$$\widehat{f}(m) := \frac{1}{\sqrt{N^d}} \sum_{x \in \mathbb{Z}_N^d} f(x)\chi(-m \cdot x),$$

where $\chi(x) = e^{2\pi i x/N}$. Then, the inverse transform formula follows:

$$f(x) = \frac{1}{\sqrt{N^d}} \sum_{m \in \mathbb{Z}_N^d} \widehat{f}(m)\chi(m \cdot x).$$

Fourier Transform Notation

- We will call an arbitrary function $f : \mathbb{Z}_N^d \to \mathbb{C}$ a **signal**.
- We will call an arbitrary function's Fourier transform $\widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ a **frequency**.

Uncertainty Principle

- Heisenberg's uncertainty principle:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2},$$

  where σ_x, σ_p are the standard deviations of position and momentum, respectively.
- Position and momentum are Fourier conjugates!

Uncertainty Principle

Theorem (Uncertainty Principle)

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a nonzero function with support $\operatorname{supp}(f) \subseteq \mathbb{Z}_N^d$. Let $\widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ denote the discrete Fourier transform of f, with support $\operatorname{supp}(\widehat{f}) \subseteq \mathbb{Z}_N^d$. Then the following inequality holds:

$$|\operatorname{supp}(f)| \cdot |\operatorname{supp}(\widehat{f})| \geq N^d.$$

Discrete L_1-norm

Definition (L_1 Norm)

For a function $f : \mathbb{Z}_N^d \to \mathbb{C}$, the L_1 norm is defined as:

$$\|f\|_{L_1(\mathbb{Z}_N^d)} := \sum_{x \in \mathbb{Z}_N^d} |f(x)|$$

Unique Recovery Principle

Theorem (Classical Recovery Condition [DS89])

Let $f : \mathbb{Z}_N \to \mathbb{C}$ such that $\widehat{f}$ is supported in $E \subset \mathbb{Z}_N$. Suppose that f is transmitted but the values $\{f(x)\}_{x \in M}$ are unobserved, where $M \subset \mathbb{Z}_N$, with

$$|E| \cdot |M| < \frac{N}{2}. \tag{1}$$

Then f can be recovered exactly and uniquely. Moreover,

$$f = \arg\min_g \|\widehat{g}\|_{L^1(\mathbb{Z}_N)} \tag{2}$$

with the constraint $f(x) = g(x)$ for $x \notin M$.

Gabor Transform

What is the Gabor transform?

Definition (General Gabor Transform)

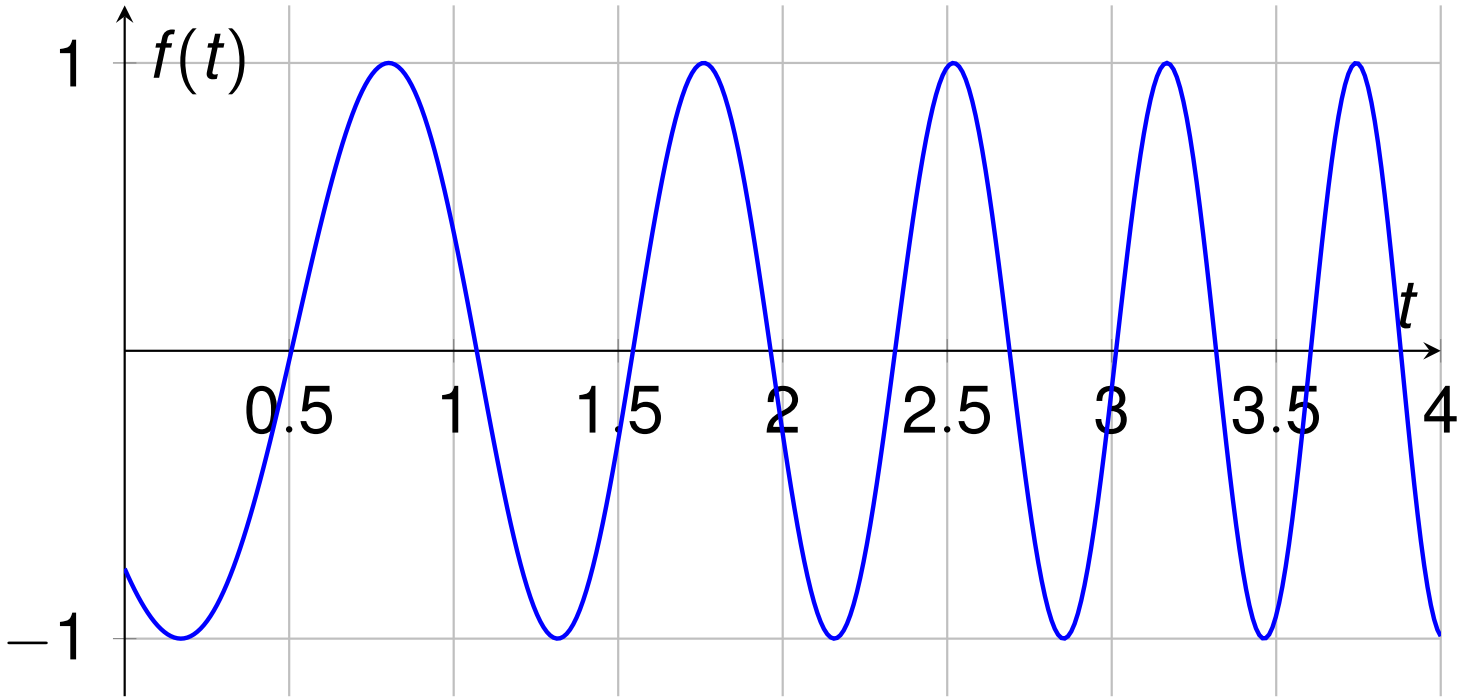
The Gabor transform of a signal $f(t)$ is defined by

$$G_f(\omega, \tau) = \int_{-\infty}^{\infty} f(t)g(t-\tau)e^{-i\omega t}dt,$$

where g is a window function, commonly taken to be $g(t) = \exp(-\pi t^2)$.

What is the Gabor transform?

Analyzing local sections of a signal as it changes over time:



The discrete Gabor transform

Definition (Discrete Gabor Transform)

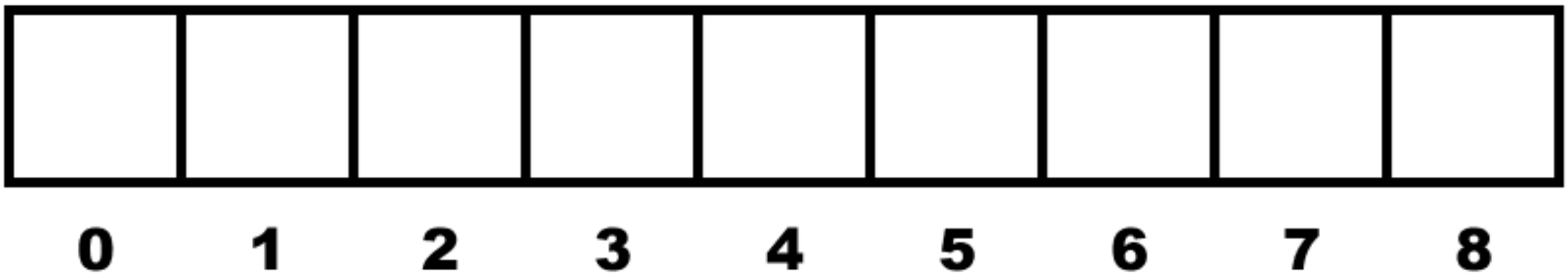
For a function $f : \mathbb{Z}_N^d \to \mathbb{C}$, the normalized discrete Gabor transform is the function

$$Gf(m, \xi) := \frac{1}{\sqrt{N^d}} \sum_{x \in \mathbb{Z}_N^d} f(x) g(x - \xi) \chi(-m \cdot x),$$

where $\chi(x) = e^{2\pi i x/N}$, and g is a window function.

An example

Consider a function $f : \mathbb{Z}_9 \to \mathbb{C}$:

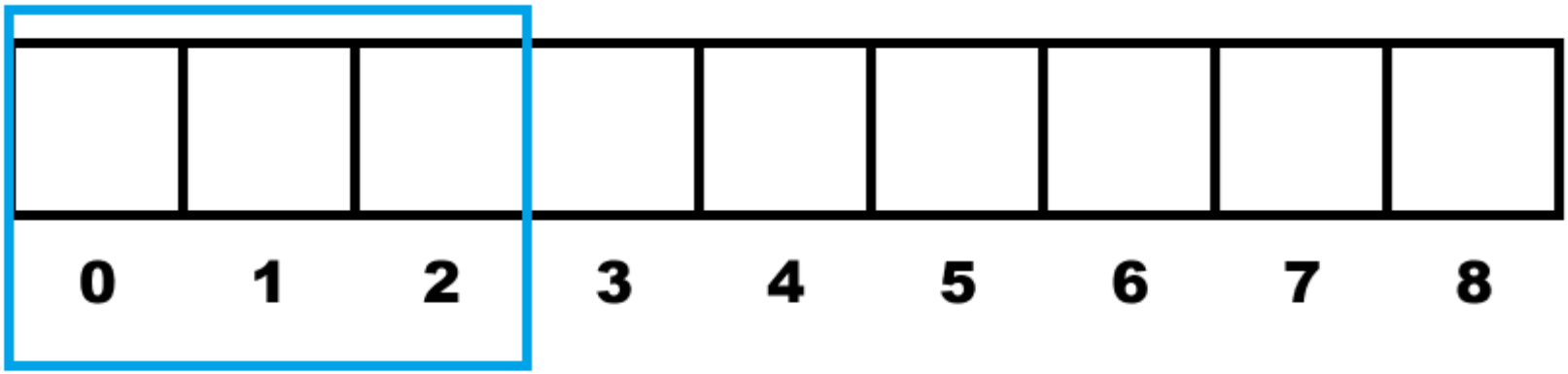


An example

A simple window function is

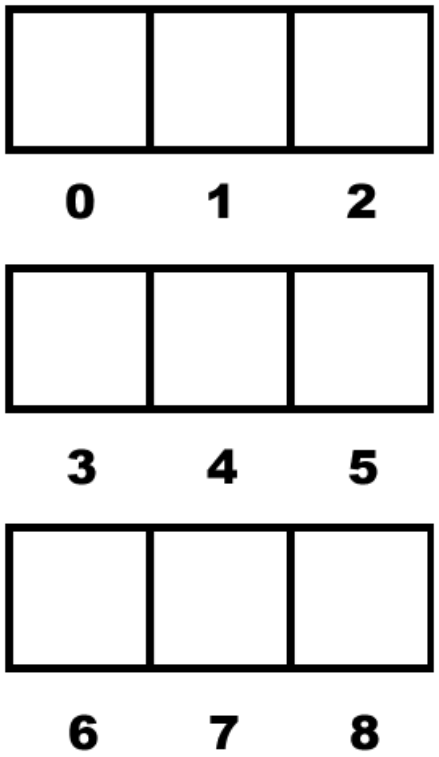
$$g(x) = \begin{cases} 1 & \text{if } x = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $Gf(1, m)$ is the Fourier transform of the first 3 terms.



An example

Another way to organize the information:



Gabor transform along a row

Definition (Gabor Transform Along a Row)

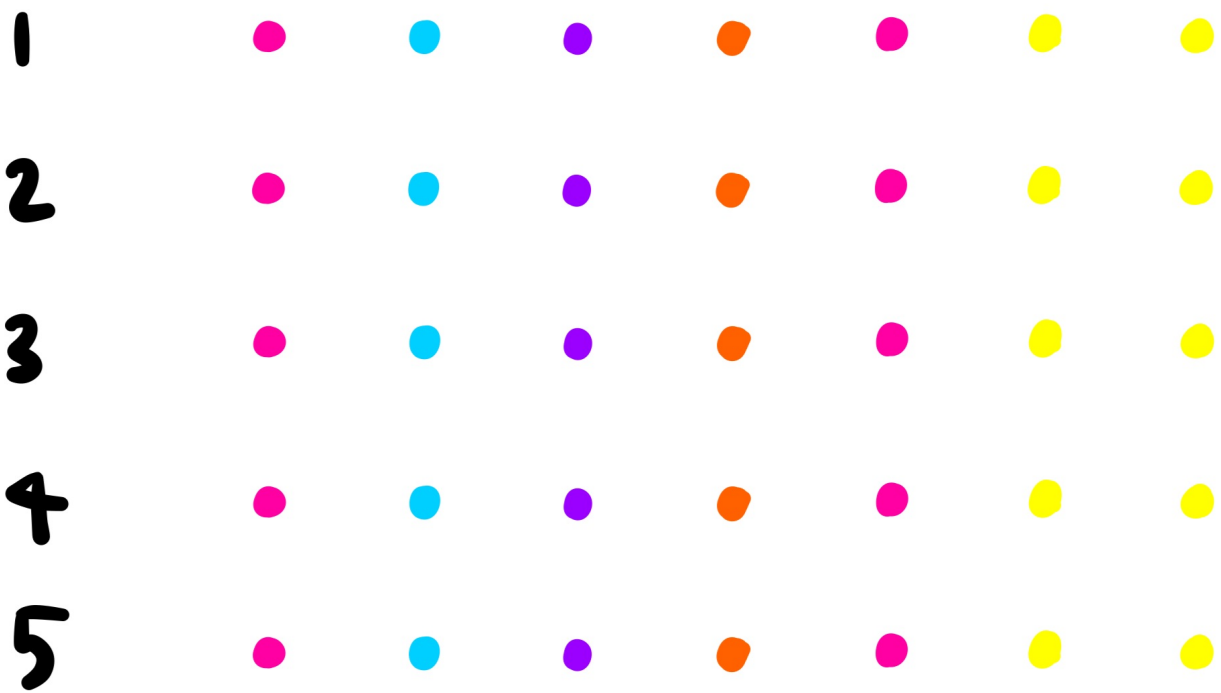
Given a function $f : \mathbb{Z}_N \times \mathbb{Z}_T \to \mathbb{C}$, we define its row-wise Gabor Transform $Gf : \mathbb{Z}_N \times \mathbb{Z}_T \to \mathbb{C}$ by

$$Gf(m, a) := N^{-1/2} \sum_{t \in \mathbb{Z}_N} f(t, a) e^{-2\pi i m \cdot t},$$

i.e., $Gf(m, a) := \widehat{f(\text{—}, a)}(m)$.
We thus have the inverse Gabor transform given by

$$f(t, a) = N^{-1/2} \sum_{m \in \mathbb{Z}_N} Gf(m, a) e^{2\pi i t \cdot m}$$

What happens if we send a signal multiple times?



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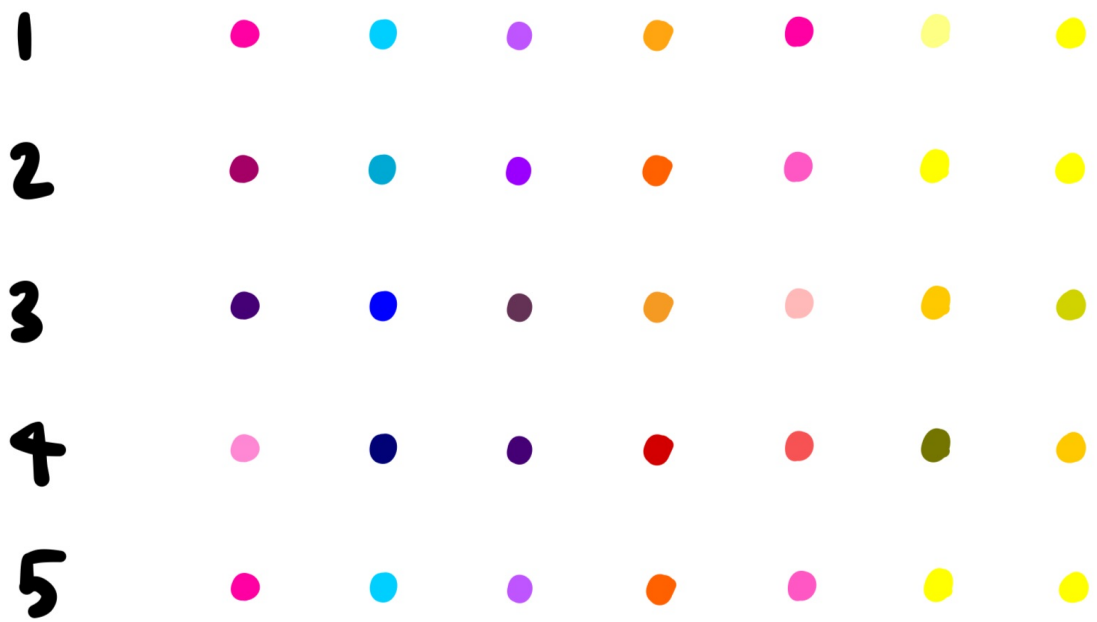

What happens if we send a signal multiple times?

Building on the work of Burstein et al '25 [BIMN25]:

Theorem (SMALL 2025; set-up)

Suppose $f : \mathbb{Z}_N \times \mathbb{Z}_T \to \mathbb{C}$, *we transmit* f, *the values in* $M \times \mathbb{Z}_T$ *are unobserved,* Gf *has support in* $S \times \mathbb{Z}_T$, $2|M||S| < N$, *and for all* $t \notin M$,

$$\sum_{c,b,\in A^c} |f(t,c) - f(t,b)| \leq \frac{\varepsilon}{T^2} \sum_{c,b\in\mathbb{Z}_T} |f(t,c) - f(t,b)| \qquad (3)$$

where $A \subseteq \mathbb{Z}_T$ *and* $0 < \varepsilon < 1$.

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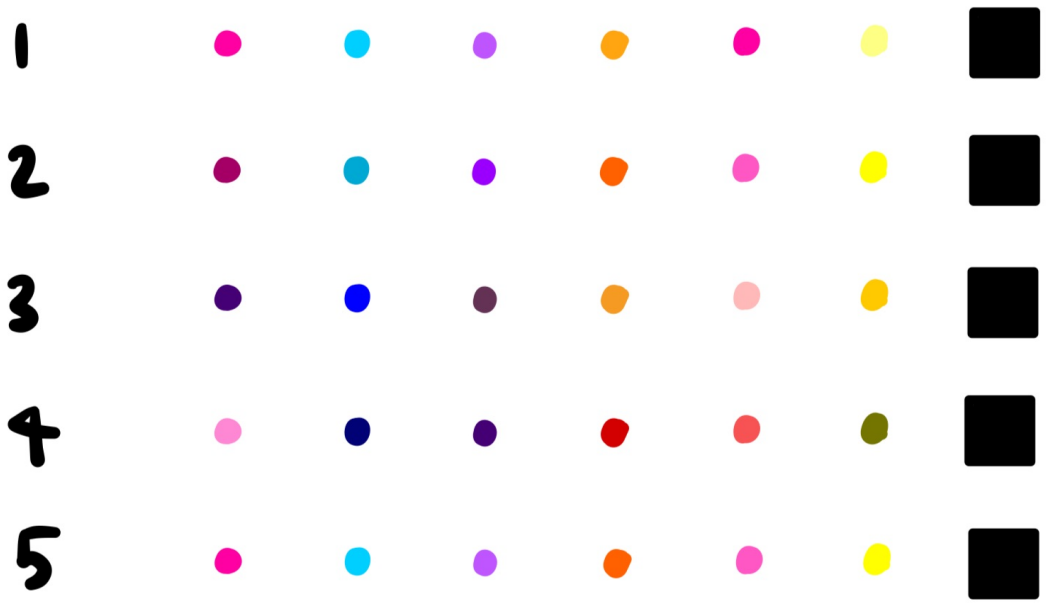

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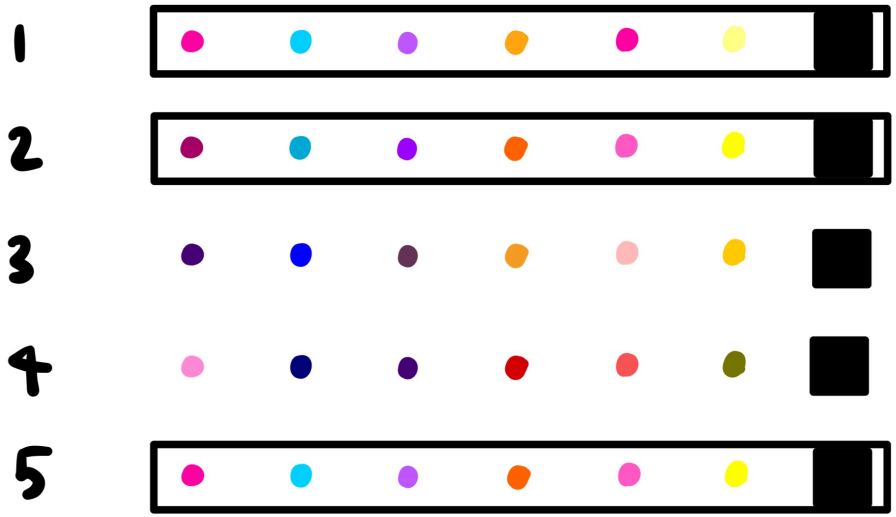

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What happens if we send a signal multiple times?

Theorem (SMALL 2025; continued)

Define $g := \text{argmin}_u \left\|\widehat{u}\right\|_{L^1(\mathbb{Z}_N)}$, *with the constraint* $u(t) = f(t,a)$ *for* $t \notin M$, *where* $a \in A^c$ *minimizes*

$$\sum_{t\in M^c}\sum_{b\in A^c} |f(t,a) - f(t,b)| \tag{6}$$

and let

$$h(t) := \left(\frac{1}{|A^c|}\sum_{b\in A^c} f(t,b)\right) - g(t). \tag{7}$$

Then,

$$\frac{1}{|M|}\left\|h\right\|_{L^1(M)} \leq \frac{3\varepsilon|S|}{|A^c|^2T^2(N-2|M||S|)}\sum_{t\in M^c}\sum_{b,c\in\mathbb{Z}_t} |f(t,b) - f(t,c)|. \tag{8}$$

Sending lines in $\mathbb{Z}_p^2$

- What if instead of sending the Fourier transform of each row, we send the Fourier transform of f restricted to lines in $\mathbb{Z}_N^d$?

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- Since the subgroups of $\mathbb{Z}_p^2$ are easier to think about than subgroups of $\mathbb{Z}_N^d$ for general N, d, we'll start by considering $f : \mathbb{Z}_p^2 \to \mathbb{C}$.

Sending lines in $\mathbb{Z}_p^2$

- $\mathbb{Z}_p^2$ has $p+1$ subgroups of order p, which intersect only at $(0,0)$. We can list these subgroups $H_0, ..., H_p$ as follows:

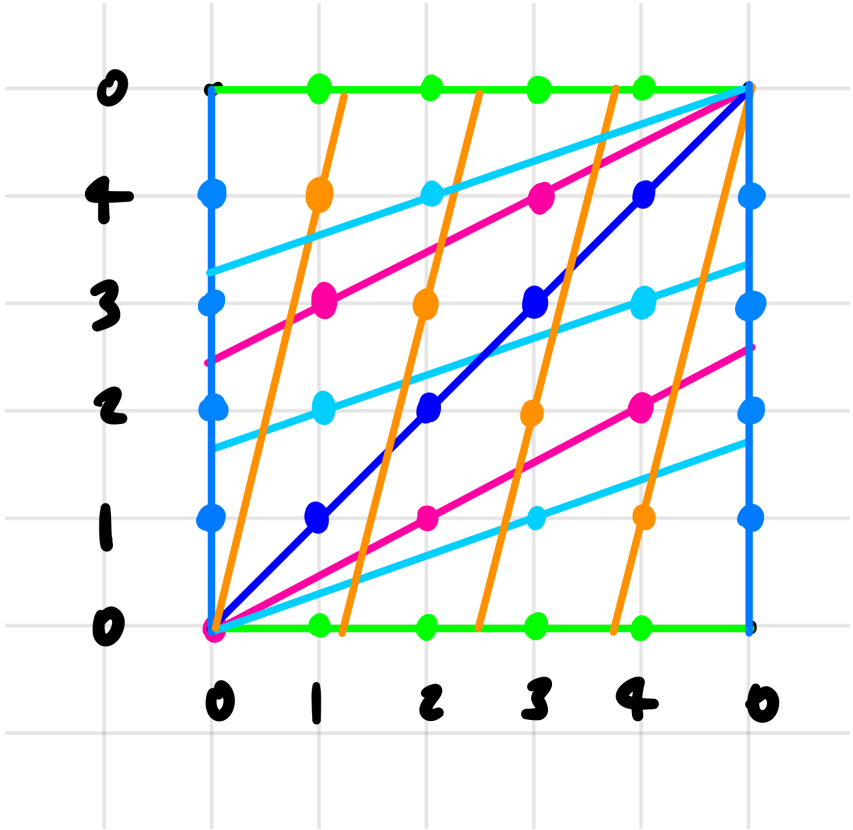
$$H_0 := \{t(0,1) : t \in \mathbb{Z}_p\}$$

$$H_j := \{t(1,j) : t \in \mathbb{Z}_p\}$$

for $1 \leq j \leq p$.

Sending lines in $\mathbb{Z}_p^2$

$\mathbb{Z}_5^2$:



Sending lines in $\mathbb{Z}_p^2$

Definition (Gabor transform along a line)

Suppose $f : \mathbb{Z}_p^2 \to \mathbb{C}$, where p is a prime. Define the Gabor transform of f, $Gf_k : \mathbb{Z}_p^2 \to \mathbb{C}$, along the line $H_k \subseteq \mathbb{Z}_p^2$ by

$$Gf_k(m,n) := \frac{1}{p}\sum_{x=0}^{p-1} f(ax,bx)\chi(x(am+bn))$$

where $H_k = \{(at,bt) : t \in \mathbb{Z}_p\}$ and $\chi(t) := \exp\left(\frac{2\pi it}{p}\right)$.

Sending lines in $\mathbb{Z}_p^2$

- Note:

$$\widehat{f}(m,n) = \frac{1}{p} \sum_{(x,y)\in\mathbb{Z}_p^2} f(x,y)\chi(mx+ny)$$

So,

$$\widehat{f}(m,n) = \left(\sum_{k=0}^{p} Gf_k(m,n) \right) - \frac{p}{p} f(0,0)$$

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Thus, if we assume $f(0,0) = 0$, we have:

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Sending lines in $\mathbb{Z}_p^2$

- Rather than thinking of $Gf_k(m, n)$ as a function on the whole grid $\mathbb{Z}_p^2$, it turns out we can think of it as a function $\widetilde{Gf_k} : \mathbb{Z}_p \to \mathbb{C}$.

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- Assuming $f(0,0) = 0$, this means that we can transmit $Gf_k(m,n)$ for all k, m, and n (and thus also $\widehat{f}(m,n) = \sum_{k=0}^{p} Gf_k(m,n)$), while only transmitting $p(p+1)$ values.

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- Can we get better recovery if we send this rather than the Fourier transform of f over the entire grid?

Sending lines in $\mathbb{Z}_p^2$

Proposition (SMALL 2025)

$f : \mathbb{Z}_p^2 \to \mathbb{C}$ with $f(0,0) = 0$. We transmit $\widetilde{Gf}_k$ for $0 \leq k \leq p$. Let E_k denote the support of $f|_{H_k}$ and define $E := \bigsqcup_{k=0}^{p} E_k$. Let M_k denote the values of $\widetilde{Gf}_k$ that are missing and define $M := \bigsqcup_{k=0}^{p} M_k$.

If $|E_k| = \dfrac{|E|}{p+1}$ and $|M_k| = \dfrac{|M|}{p+1}$ for all k, then if $|E||M| < \dfrac{p(p+1)^2}{2}$, we can uniquely recover f.

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- Classical recovery condition: $|E||M| < \dfrac{p^2}{2}$.

Future Works

Closing

Future Works

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- $\widehat{f}(m,n) = \sum_{k=0}^{p} Gf_k(m,n)$ means that we can transmit $\widehat{f}$ without directly transmitting $\widehat{f}$ itself. If some of the $Gf_k(m,n)$'s are missing, though, can we still piece together what some of the $\widehat{f}(m,n)$'s are?

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- General $\mathbb{Z}_N^d$.

Future Works: Spherical Harmonics

- The spherical harmonics $Y_l^m(\theta, \phi) : S^2 \to \mathbb{C}$ are such that $f(r, \theta, \phi) = R(r)Y(\theta, \phi)$ satisfies $\nabla^2 f = 0$. These Y_l^m form an orthonormal basis for functions defined on a sphere, so we have the spherical harmonic series

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m Y_l^m(\theta, \phi),$$

  where $f_l^m = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) \overline{Y_l^m(\theta, \phi)} d\theta d\phi$.
- What about the discrete case?

Acknowledgments

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- For $1 \leq k \leq p$, $(1,0) \notin H_k^\perp$, so we can represent the p disjoint cosets of $H_k^\perp$ (i.e. the elements of $\mathbb{Z}_p^2/H_k^\perp$) as $(t,0) + H_k^\perp$ for $t \in \mathbb{Z}_p$. Likewise, since $(0,t) \notin H_0^\perp$, we can represent the elements of $\mathbb{Z}_p^2/H_0^\perp$ as $(0,t) + H_0^\perp$.

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- Assuming $f(0,0) = 0$, this means that if we transmit $Gf_0(0,t)$ and $Gf_k(t,0)$ for $1 \leq k \leq p$ for $t \in \mathbb{Z}_p$, we will have transmitted $Gf_k(m,n)$ for all k and $(m,n) \in \mathbb{Z}_p^2$, and thus also $\widehat{f}(m,n) = \sum_{k=0}^{p} Gf_k(m,n)$.

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- Can we get better recovery if we send this rather than the Fourier transform of f over the entire grid?

Sending lines in $\mathbb{Z}_p^2$

- What happens if we assume $f(0,0) = 0$ and transmit $Gf_0(0,t)$ and $Gf_k(t,0)$ for $1 \leq k \leq p$ for $t \in \mathbb{Z}_p$, but now some of the values we transmit get lost?

Sending lines in $\mathbb{Z}_p^2$

Definition ($H_k^\perp$)

Let $(a, b) \neq (0, 0)$ be such that $H_k = \{(at, bt) : t \in \mathbb{Z}_p\}$.
Define $H_k^\perp \subseteq \mathbb{Z}_p^2$ by

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- $H_k^\perp$ is a subspace of $\mathbb{Z}_p^2$ of dimension 1. Thus, for each k, $\mathbb{Z}_p^2 / H_k^\perp$ is a vector space of dimension $2 - 1 = 1$.

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Let $(a, b) \neq (0, 0)$ be such that $H_k = \{(at, bt) : t \in \mathbb{Z}_p\}$. Define $H_k^\perp \subseteq \mathbb{Z}_p^2$ by

$$H_k^\perp := \{(m, n) \in \mathbb{Z}_p^2 : am + bn = 0\}.$$

- $H_k^\perp$ is a subspace of $\mathbb{Z}_p^2$ of dimension 1. Thus, for each k, $\mathbb{Z}_p^2/H_k^\perp$ is a vector space of dimension $2 - 1 = 1$.
- For all $(m', n') \in H_k^\perp$,

$$Gf_k(m + m', n + n') = Gf_k(m, n).$$

What happens if we send a signal multiple times?

Theorem (Burstein et al '25 [BIMN25])

Let $f : \mathbb{Z}_N \to \mathbb{C}$, and suppose that f is transmitted but the values $\{f(x)\}_{x \in M}$ are unobserved. Suppose that

$$\left\|\widehat{f}\right\|_{L^1(S^c)} \leq \frac{\varepsilon}{N} \left\|\widehat{f}\right\|_{L^1(\mathbb{Z}_N)}. \tag{9}$$

Also suppose that $2|M||S| < N$. Let $g = \operatorname{argmin}_u \left\|\widehat{u}\right\|_{L^1(\mathbb{Z}_N)}$ with the constraint $f(x) = u(x)$ for $x \notin M$. Let $h = f - g$. Then,

$$\frac{1}{|M|} \sum_{x \in M} |h(x)| \leq \frac{2\varepsilon}{N - 2|M||S|} \sum_{x \in \mathbb{Z}_N} |f(x)|. \tag{10}$$