Congruence Classes of Simplex Structures in Finite Field Vector Spaces

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The Erdős-Falconer Distance Problem over Finite Fields

Question

For the distance function

$$||x - y|| = (x_1 - y_1)^2 + \cdots + (x_d - y_d)^2,$$

how big does a subset E of $\mathbb{F}_q^d$ need to be to determine all/positive proportion of distances in $\mathbb{F}_q^d$?

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Previous Results: $|E| \gtrsim q^s$

- Iosevich-Rudnev: $s = \frac{d+1}{2}$
- Chapman et al: $s = \frac{4}{3}$ for $d = 2$
- Murphy et al: $s = \frac{5}{4}$ for $d = 2$

Erdős-Falconer for general graphs

Definition

Given a graph G, two embeddings $p, p' : V(G) \to \mathbb{F}_q^d$ of a graph G are **congruent** if for every $(v_i, v_j) \in E(G)$,

$$||p(v_i) - p(v_j)|| = ||p'(v_i) - p'(v_j)||.$$

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- Rigid structures: Complete graphs
- Loose structures: Paths, Trees, Cycles

Rigid Structures

Embeddings of complete graph on $k+1$ vertices are k-dimensional simplices (or just k-simplices) in $\mathbb{F}_q^d$

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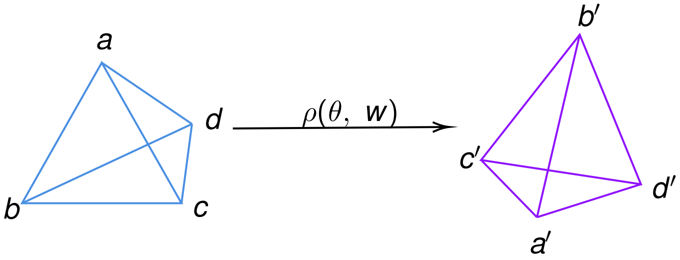
We can redefine congruence in terms of group actions and orthogonal transformations thanks to rigidity!



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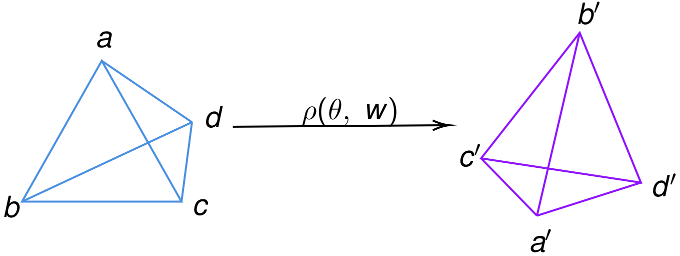

Theorem (Bennett et. al. 2013, McDonald 2019)

For $E \subset \mathbb{F}_q^d$ and $s \geq \frac{dn+1}{n+1}$, E contains a positive proportion of congruence classes of n-simplices in $\mathbb{F}_q^d$.

Rigid Structures continued

Proof idea: We want to bound

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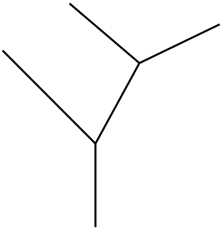
Reduce the problem to bounding a sum of the form

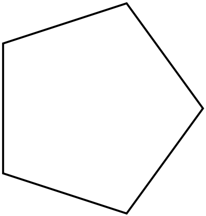
$$\sum_{\theta, w} \lambda_\theta^{n+1}(w)$$

Bound this with Fourier analytic techniques

Loose Structures

Lack of rigidity $\Rightarrow$ group actions are less effective





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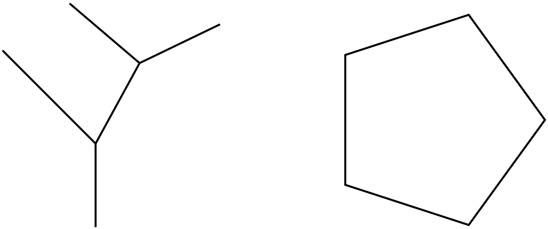


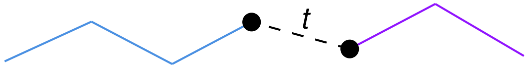
Theorem (Bennett et. al. 2018, Iosevich et. al. 2021)

For $E \subset \mathbb{F}_q^d$ and $s \geq \frac{d+1}{2}$, E contains all congruence classes of paths and trees in $\mathbb{F}_q^d$.

Important Note: this threshold is independent of the length of the path or tree!

Loose Structures

Proof idea: Fix a congruence class, and use inductive nature of graph



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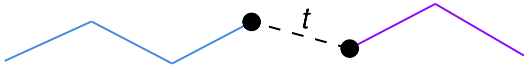


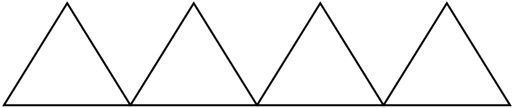
The problem reduces to proving the functional inequality

$$\sum_{x,y} f(x)g(y)S_t(x-y) = \frac{|S_t|}{q^d}||f||_1||g||_1 + \text{error}$$

and plugging in path counting functions for f and g.

Chains of Simplices

Purely inductive techniques can be used to take care of chains of simplices!



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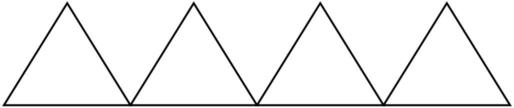

Theorem (SMALL 2023)

For $s \geq k + \frac{d-1}{2}$, E contains all congruence classes of chains of k-simplices whenever $|E| \gtrsim q^s$.

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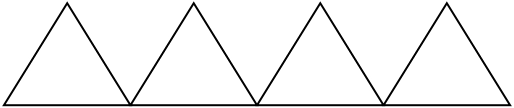

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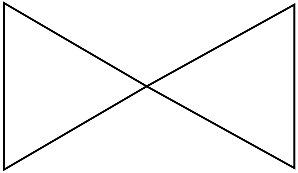
Fallback: Inductive techniques give trivial results for high-dim simplices in low-dim spaces.

The Bowtie Graph

Can group actions and inductive techniques be combined?



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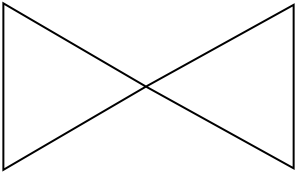
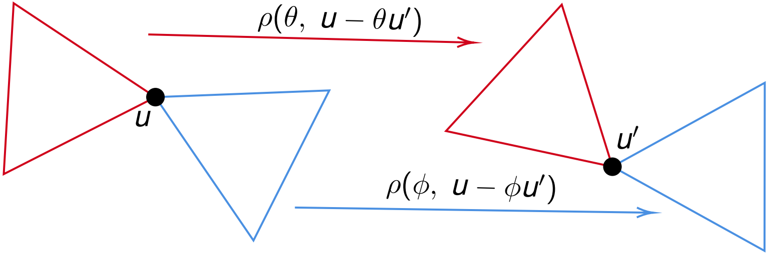


Theorem (Aksoy, Iosevich, McDonald 2024)

Let $E \subset \mathbb{F}_q^2$ and let G be the bowtie graph as pictured above. Suppose $|E| \gtrsim q^{\frac{12}{7}}$, then E determines a positive proportion of congruence classes of the bowtie graph.

Bowtie Graph Example

Proof sketch: We can take advantage of rigidity to redefine congruence classes in terms of group actions!



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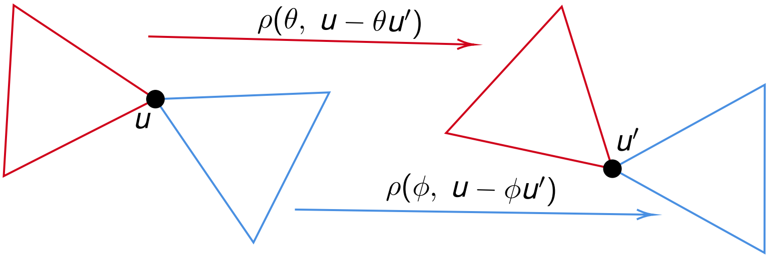

$$\#\{\text{pairs of congruent embeddings of } B \text{ in } E\}$$

$$= \sum_{\theta,\phi} \sum_{u,u'} \lambda_\theta^2(u - \theta u')\lambda_\phi^2(u - \phi u')$$

Bowtie Graph Example

Key idea: We have that

$$\sum_{\theta,\phi}\sum_{u,u'}\lambda_\theta^2(u-\theta u')\lambda_\phi^2(u-\phi u') \leq \sum_{\theta,\phi}\sum_{u,u'}\lambda_\theta^3(u-\theta u')\lambda_\phi^1(u-\phi u').$$

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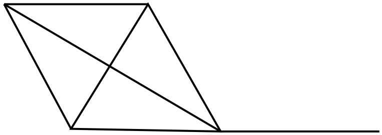
This sum corresponds to the number of pairs congruent embeddings of K in E , where K is the kite graph:



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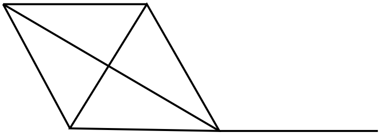

This is morally just a 3-simplex! Use Fourier analysis to separate contributions from 1 and 3 simplices, and use group actions and inductive approaches accordingly.

Bowtie Graph Example

How did we get the inequality on the previous slide?

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Theorem (Hadamard Three Lines)

Suppose $f(z) : \mathbb{C} \to \mathbb{C}$ is bounded, continuous on $\{z : \text{Re}(z) \in [a, b]\}$, and holomorphic on the interior. Then for $M(x) = \sup_{y \in \mathbb{R}} |f(x + iy)|$, the function $\log(M(x))$ is convex on $[a, b]$.

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Define $\psi(z) : \mathbb{C} \to \mathbb{C}$ as

$$\psi(z) = \sum_{\theta,\phi} \sum_{u,u'} \lambda_\theta^{2-z}(u - \theta u')\lambda_\phi^{2+z}(u - \phi u').$$

By H3L, $\log(\psi(z))$ is convex on $[-1, 1]$, so

$$\psi(0) \leq \sqrt{\psi(1)\psi(-1)}.$$

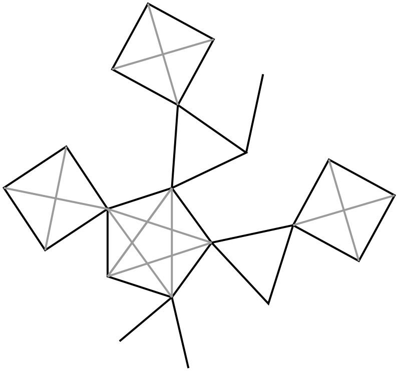
Bowtie Graph Example

Big takeaway: Hadamard 3 Lines lets us "unbalance" the powers of λ_θ, and algebraic modifications to our sum using H3L corresponds to geometric modifications of our graph!

Simplex Trees

Definition (Simplex Tree)

A simplex tree is a generalization of a tree where simplices act as vertices and shared vertices where simplices are attached act as edges.



A weak version of our main result

Theorem (SMALL 2024, weak version)

For a simplex tree $\mathcal{T}$, define

$$N = 1 + \sum_{\substack{S \in \mathcal{T} \\ \dim(S) > 1}} (\dim(S) - 1).$$

For an $E \subset \mathbb{F}_q^d$, suppose that for $s = \frac{dN+1}{N+1}$,

$$|E| \gtrsim q^s.$$

Then E contains a positive proportion of congruence classes of embeddings of $\mathcal{T}$ in $\mathbb{F}_q^d$.

Note: $s = \frac{dN+1}{N+1}$ is same s for an N-simplex.

The Game Plan

The problem immediately reduces to bounding the number of pairs of congruent embeddings of $\mathcal{T}$ in E.

- **Step 1:** Use group actions to redefine congruence and rewrite sum in a more workable form

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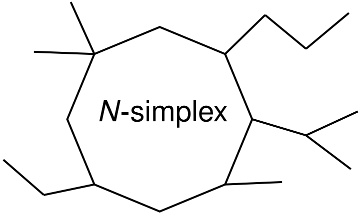

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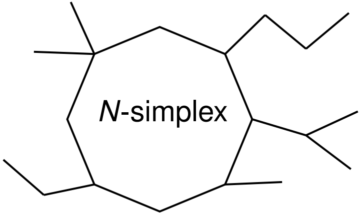
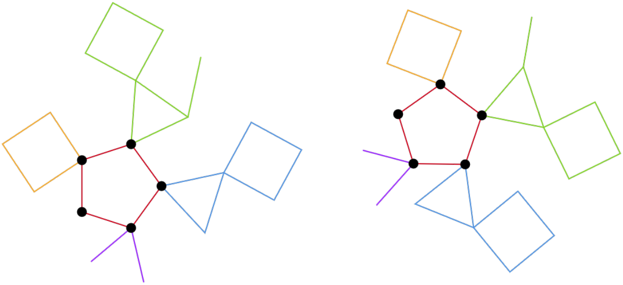


- **Step 3:** Use Fourier analytic techniques to separate contributions of large simplex and N-simplex trees from final sum

Step 1 Overview

Step 1: Use group actions to redefine congruence and write an expression for the number of pairs of congruent embeddings of $\mathcal{T}$ in E.



$h_1 : V(\mathcal{T}) \to \mathbb{F}_q^d$ $h_2 : V(\mathcal{T}) \to \mathbb{F}_q^d$

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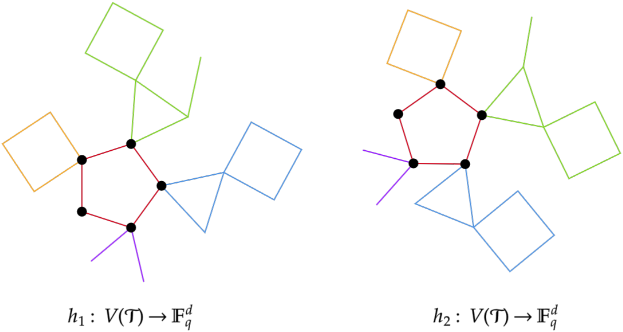
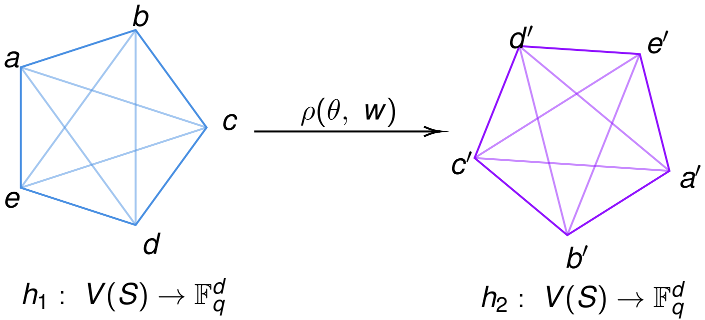


Idea: Count how many pairs of congruent embeddings of the root simplex extend to congruent embeddings of $\mathcal{T}$.

The root simplex

We want to count the number of pairs of congruent embeddings of the root simplex.
Again, we define

$$\lambda_\theta(w) = \#\{u, u' \in E : u' - \theta u = w\}.$$



The root simplex

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We sum over all θ, and (modulo technicalities) our final count is

$$\#\{\text{pairs of congruent embeddings of root simplex in } E\}$$

$$= \sum_{\theta \in O_d(\mathbb{F}_q)} \lambda_\theta^5(w).$$

The whole simplex tree

Idea: Count how many pairs of congruent embeddings of the root simplex extend to congruent embeddings of $\mathcal{T}$.

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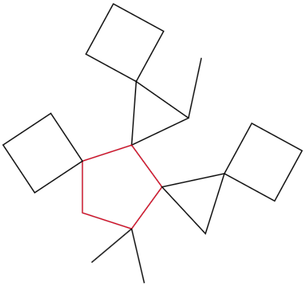

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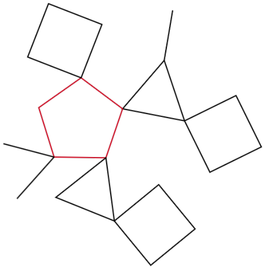

$h_2 : V(\mathcal{T}) \to \mathbb{F}_q^d$

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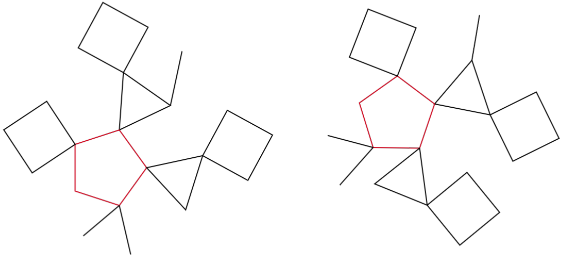
$$\sum_{w \in \mathbb{F}_q^d} \sum_{\theta \in O_d(\mathbb{F}_q)} \prod_{i=1}^{5} \sum_{\substack{x_i, x_i' \in E \\ x_i' - \theta x_i = w}} 1$$



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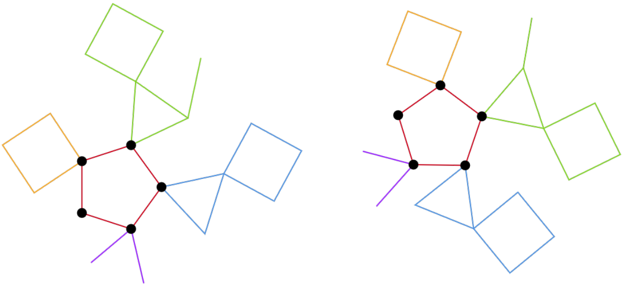

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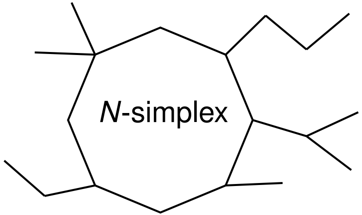
Step 2

Step 2: Use H3L to reduce to finding an s for a simpler class of simplex trees

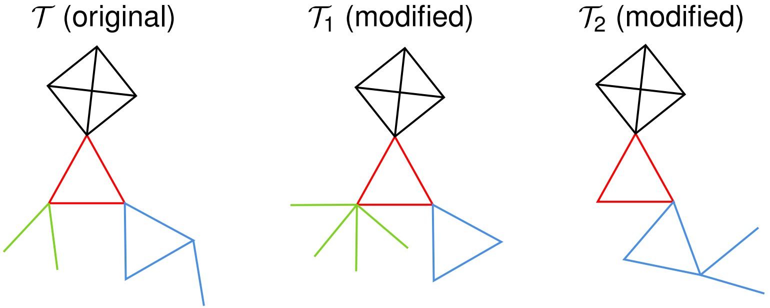


Two main geometric operations:

- Branch Shifting
- Simplex Unbalancing

Branch Shifting

For any two vertices of a simplex tree $\mathcal{T}$ and their corresponding branches, we can remove one branch and duplicate the other giving two new trees $\mathcal{T}_1, \mathcal{T}_2$.



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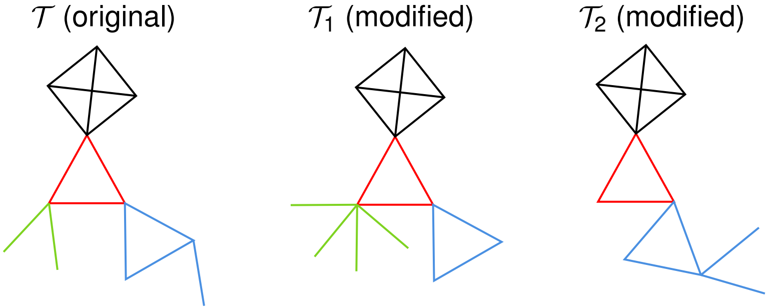
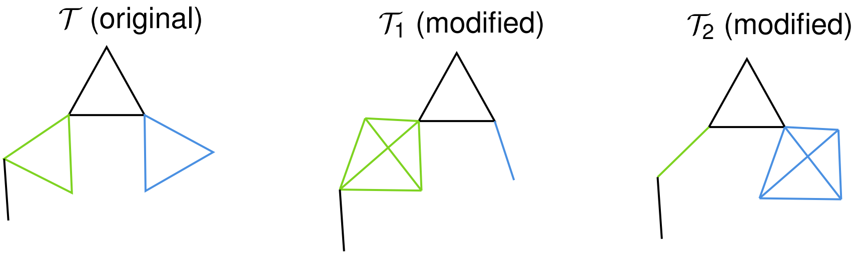


Lemma (Branch shifting)

To find an s for $\mathcal{T}$, it suffices to find an s for $\mathcal{T}_1$ and $\mathcal{T}_2$.

Simplex Unbalancing

For any two simplices of the simplex tree $\mathcal{T}$ we can move some number of free vertices from the first simplex to the other creating the graph $\mathcal{T}_1$ or we can move some free vertices from the second simplex to the first creating $\mathcal{T}_2$.



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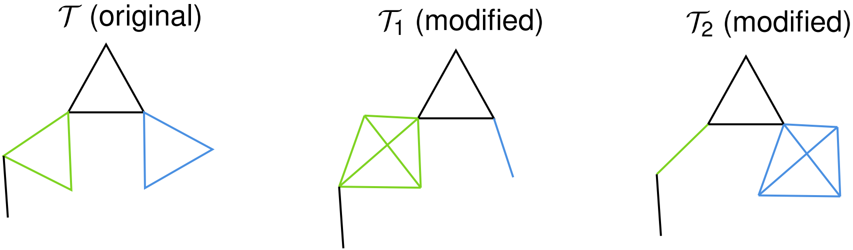

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Step 2: Reduction to a single simplex

We use an algorithm to reduce our simplex tree into one large simplex and a tree of 1-leafs.

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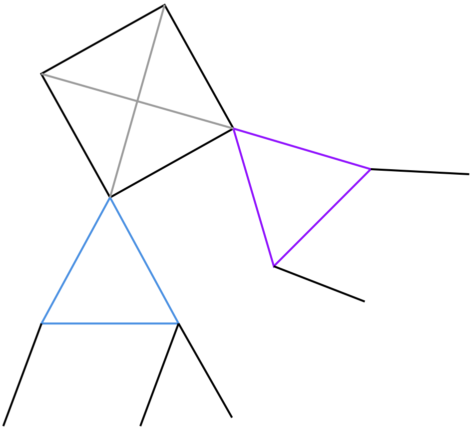
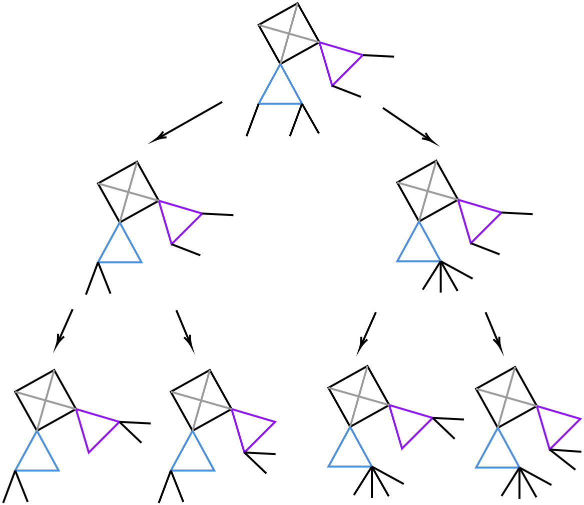


Step 2: Reduction to a single simplex

First apply Branch Shifting



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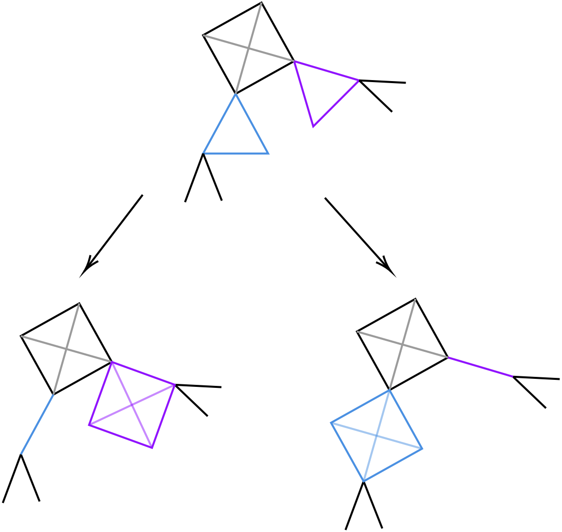
Then apply Simplex Unblancing



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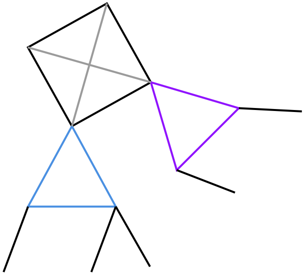
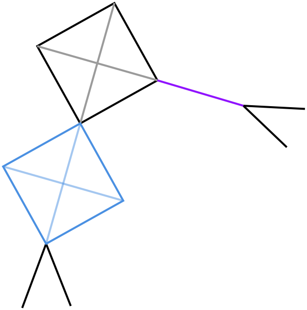
Before



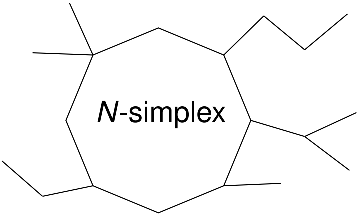
After



Step 2: Reduction to a single simplex

This leaves us with an N-simplex for

$$N = 1 + \sum_{\substack{S \in \mathcal{T} \\ \dim(S) > 1}} (\dim(S) - 1)$$



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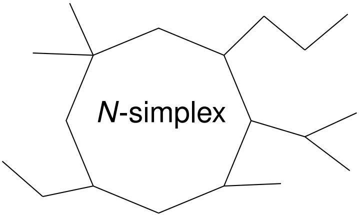

We're left to find an s for this simple class of simplex structures.

Step 3: Fourier Analysis

For remaining sum, need to separate contributions from N-simplex and contributions from the paths/trees using Fourier analysis.

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The highlights:

- For counting functions like λ_θ, we often have $\hat{\lambda}_\theta(0)$ large, so we split

$$\sum_m \widehat{\lambda}_\theta(m) = \widehat{\lambda}_\theta(0) + \sum_{m \neq 0} \widehat{\lambda}_\theta(0)$$

- The bounds we have for paths and trees are independent of length and structure

Our (actual) main result

Theorem (SMALL 2024, strong version)

For a simplex tree $\mathcal{T}$, for $k < \frac{d+1}{2}$, define

$$N_k = k + \sum_{\substack{S \in \mathcal{T} \\ \dim(S) > k}} (\dim(S) - k).$$

For an $E \subset \mathbb{F}_q^d$, suppose for $s = \max\left(\frac{dN_k+1}{N_k+1},\ k + \frac{d-1}{2}\right)$,

$$|E| \gtrsim q^s.$$

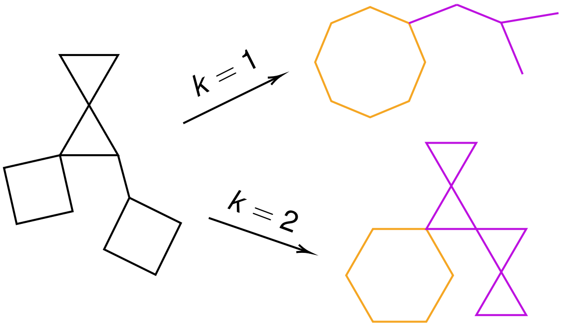
Then E contains a positive proportion of congruence classes of embeddings of $\mathcal{T}$ in $\mathbb{F}_q^d$.

Note: We do better in $d = 2$ for technical reasons!

What is this k parameter?

Changing k allows us to scale between group-action and inductive techniques!

Idea: We instead use Hadamard Three Lines to reduce to a central simplex with k-simplex trees attached, so central simplex is smaller.



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Theorem (SMALL 2023)

For $s \geq k + \frac{d-1}{2}$, E contains all congruence classes of **a certain class of simplex trees of k-simplices** whenever $|E| \gtrsim q^s$.

Note: This class is much smaller than the ones we handle, but we can modify branches of large simplex to fit this class!

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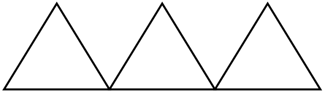
As these bounds are independent of the size of the simplex tree, we can bound terms corresponding to these trees with Fourier analysis.

Applications

Our results give us

- An improvement to results of SMALL 2023 for many chains and trees of simplices



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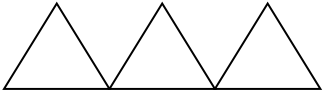

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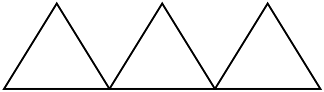

$$s = \frac{17}{5} \textit{ in } \mathbb{F}_q^4, \textit{ compare to } \frac{11}{2}$$

- New nontrivial results for all simplex trees in $d = 2$
- An extension of SMALL 2023's work to our full class of simplex trees, with same value of s

Corollary

For $s \geq k + \frac{d-1}{2}$, E contains all congruence classes of **any** k**-simplex tree** whenever $|E| \gtrsim q^s$.

Applications

Theorem (Iosevich-Parshall 2019)

In $\mathbb{F}_q^d$, for any graph G with maximum vertex degree t, let $s = t + \frac{d-1}{2}$. If $|E| \gtrsim q^s$ then E contains all congruence classes of G in $\mathbb{F}_q^d$

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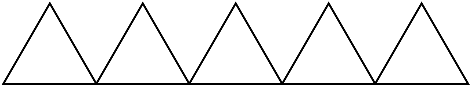
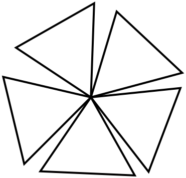
Note: Our result is independent of vertex degree.





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It is also Joe's birthday! Happy birthday Joe!!!

Thank you!

We dedicate this talk to Hadamard's 158.5th birthday.

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