On Zeckendorf Related Partitions Using the Lucas Sequence

David C. Luo (joint work with Hùng V. Chu and Steven J. Miller)

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Fibonacci and Lucas Number Sequences

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Definition 1

We define the *Fibonacci sequence* $\{F_k\}_{k=0}^{\infty}$ by $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-2} + F_{k-1}$ for $k \geq 2$. We define the *Lucas sequence* $\{L_k\}_{k=0}^{\infty}$ by $L_0 = 2$, $L_1 = 1$, and $L_k = L_{k-2} + L_{k-1}$ for $k \geq 2$.

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Definition 2

A *partition* of a positive integer n is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition.

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Theorem 4 (Zeckendorf)

Every natural number can be partitioned into the sum of distinct and non-consecutive terms of the Lucas sequence [3].

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Example

5 is uniquely partitioned into F_5 in $\{F_2, F_3, \ldots\}$. Its partition is not unique in the Lucas sequence as

$$\begin{aligned} 5 &= L_0 + L_2 = 2 + 3 \\ &= L_1 + L_3 = 1 + 4. \end{aligned}$$

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12 is uniquely partitioned into $F_2 + F_4 + F_6 = 1 + 3 + 8$ in $\{F_2, F_3, \ldots\}$. Its partition is not unique in the Lucas sequence as

$$\begin{aligned} 12 &= L_1 + L_5 = 1 + 11 \\ &= L_0 + L_2 + L_4 = 2 + 3 + 7. \end{aligned}$$

Proportion of Non-uniqueness

Let $c(N)$ count the number of natural numbers that are not uniquely written as the sum of distinct and non-consecutive terms of the Lucas sequence and are at most N. Further, let $\alpha(N) = \frac{c(N)}{N}$.

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10^5	17,082	17.082%
10^6	170,820	17.082%

What is $\lim_{N\to\infty} \alpha(N)$?

Preliminaries

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Definition 5

Let $A = \{a_0, a_1, \ldots, a_m\}$ be the set consisting of the first $m+1$ terms of the sequence $\{a_k\}_{k=0}^{\infty}$. We say a sum S is a *non-consecutive sum* of A if S is the sum of distinct elements of A that are pairwise non-consecutive in $\{a_k\}_{k=0}^{\infty}$.

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Lemma 7

If $m \geq 0$, then $L_{2m+1} + 1$ has exactly two non-consecutive partitions in the Lucas sequence.

Theorem 8

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Theorem 8 (Chu, Luo, Miller)

A natural number can have at most two distinct non-consecutive partitions in the Lucas sequence.

Proof of Theorem 8

- It suffices to show that for every nonnegative integer m, there is no natural number that is equal to three or more distinct non-consecutive sums of $\{L_0, L_1, \ldots, L_m\}$.

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- No natural is equal to three or more distinct non-consecutive sums of $\{L_0\}$ and $\{L_0, L_1\}$. This shows the base case.
- Assume Theorem 8 holds for all nonnegative integers less than or equal to $m = k$. In our first case, suppose that k is odd.
- From Lemma 6, the non-consecutive sums that we can form from $\{L_0, L_1, \ldots, L_k\}$ are zero to $L_{k+1} - 1$ inclusive. When we add the term L_{k+1} to $\{L_0, L_1, \ldots, L_k\}$, all new non-consecutive sums are at least L_{k+1}. This implies there is no way to form a third distinct non-consecutive sum of $\{L_0, L_1, \ldots, L_{k+1}\}$ as there is no intersection between the non-consecutive sums in which we can form before and after the addition of the term L_{k+1}.

Proof of Theorem 8 Continued

- When $k \geq 2$ is even, we have from Lemma 6 that all non-consecutive sums we can form from $\{L_0, L_1, \ldots, L_k\}$ are the values from zero to $L_{k+1} + 1$ inclusive excluding L_{k+1}.

Proof of Theorem 8 Continued

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- When we add the term L_{k+1} to $\{L_0, L_1, \ldots, L_k\}$, all new non-consecutive sums that can be formed are at least L_{k+1} with $L_{k+1} + 1$ being the only non-consecutive sum formed again, namely $L_{k+1} + L_1$.

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- When we add the term L_{k+1} to $\{L_0, L_1, \ldots, L_k\}$, all new non-consecutive sums that can be formed are at least L_{k+1} with $L_{k+1} + 1$ being the only non-consecutive sum formed again, namely $L_{k+1} + L_1$.
- By Lemma 7, we know that $L_{k+1} + 1$ has exactly two distinct non-consecutive partitions in the Lucas sequence.
- Therefore, there is no possible way in which we can form a third distinct non-consecutive sum of $\{L_0, L_1, \ldots, L_{k+1}\}$ for any natural number. This completes the inductive step. $\square$

Proportion of Non-uniqueness

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Theorem 9 (Chu, Luo, Miller)

The proportion of natural numbers that cannot be uniquely partitioned into the sum of distinct and non-consecutive terms of the Lucas sequence converges to

$$\alpha = \frac{1}{3\Phi + 1}.$$

Brown's Criterion for Uniqueness

Theorem 10 (Brown)

Let n be a natural number. Then n can be uniquely represented as

$$n = \sum_{i=0}^{\infty} \alpha_i L_i,$$

where each α_i is a binary digit such that

(i) $\alpha_i \alpha_{i+1} = 0$ *for* $i \geq 0$ *and*

(ii) $\alpha_0 \alpha_2 = 0$ *[1].*

The Golden String

Definition 11

The *golden string* $S_\infty = BABBABABBABBA\ldots$ is defined to be the infinite string of A's and B's constructed recursively as follows. Let $S_1 = A$ and $S_2 = B$, and then, for $k \geq 3$, S_k is the concatenation of S_{k-1} and S_{k-2}, which we denote by $S_{k-1} \circ S_{k-2}$ [2].

Property 0.1

The number of B's among the first n characters of S_∞ is given by $\left\lfloor \frac{n+1}{\Phi} \right\rfloor$ [2].

Definition 12

Let K denote the set of all natural numbers who have a Lucas partition with both L_0 and L_2. Let $\{q(j)\}_{j\geq 1}$ be the strictly increasing sequence obtained by rearranging the elements of K into ascending numerical order.

Lemma 13

For $j \geq 1$, we have

$$q(j+1) - q(j) \;=\; \begin{cases} L_3, & \text{if } A \text{ is the } j\text{th character of } S_\infty, \\ L_4, & \text{if } B \text{ is the } j\text{th character of } S_\infty. \end{cases}$$

Proof of Theorem 9

- We need to find a formula for $c(N)$ in terms of N.

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- Via Lemma 13, we can write K as

$$K = \{L_0 + L_2 + a(n)L_3 + b(n)L_4 : n \geq 0\},$$

where $a(n)$ and $b(n)$ denote the number of A's and B's, respectively, among the first n characters in the golden string.

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- Hence, $c(N) = \frac{N-1}{4+\frac{3}{\Phi}}$ with an error of at most one. Therefore,

$$\lim_{N\to\infty} \frac{c(N)}{N} = \frac{1}{4+\frac{3}{\Phi}} = \frac{1}{3\Phi+1}. \qquad \square$$

Future Work

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Definition 14

An arbitrary sequence $\left\{a_k\right\}_{k=0}^{\infty}$ of natural numbers is *complete* if, and only if, every natural number n can be represented in the form

$$n = \sum_{i=0}^{\infty} c_i a_i$$

where each c_i is a binary number [1].

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- We can alter the definition of a complete sequence so that it includes the property $c_i c_{i+1} = 0$ for all i.

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