Short-Range and Random Differences in the Number of Summands of Zeckendorf Decompositions

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Joint work with Wyatt Milgrim, Steven J. Miller, Prakod Ngamlamai

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What happens to Zeckendorf Decompositions when we add them?

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Zeckendorf Decomposition

$F_1 = 1, F_2 = 2, \ldots$ are the Fibonacci numbers.

Zeckendorf's Theorem

Every non-negative integer can be uniquely written as a sum of non-adjacent Fibonacci numbers.

Compare to:

Binary Decomposition

Every non-negative number can be uniquely written as a sum of powers of 2.

Analogy

- Base 2

  non-negative integer $\longleftrightarrow$ string of 0's and 1's

- Zeckendorf

  non-negative integer $\longleftrightarrow$ string of 0's and 1's w/o consecutive 1's

Example (Zeckendorf Decoposition)

Binary

$$14 = 2^3 + 2^2 + 2^1 \quad \longleftrightarrow \quad \begin{array}{ccccccc} \cdots & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ \hline \cdots & 0 & 0 & 1 & 1 & 1 & 0 \end{array}$$

Zeckendorf

$$14 = F_6 + F_1 \quad \longleftrightarrow \quad \begin{array}{ccccccc} \cdots & F_6 & F_5 & F_4 & F_3 & F_2 & F_1 \\ \hline \cdots & 1 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Example (Adding Zeckendorf Decompositions)

We write

$$14 = (100001)_F.$$

We also define

$$Z(X) := \#\text{ summands in Zeckendorf decomposition of } X,$$

and

$$Z_i(X) := \begin{cases} 1, & \text{if } F_i \text{ is in Zeckendorf decomposition of } X \\ 0, & \text{otherwise.} \end{cases}$$

We can do arithmetic:

$$\begin{aligned} 14 + 5 = (100001)_F + (1000)_F &= (101001)_F \\ &= 19 \end{aligned}$$

Example (Adding Zeckendorf Decompositions)

Warning

It is not always the case this leads to Zeckendorf!

$$\begin{aligned} 14 + 2 = (100001)_F + (10)_F &= (100011)_F \\ &= (100100)_F \end{aligned}$$

$$\begin{aligned} 14 + 13 = (100001)_F + (100000)_F &= (200001)_F \\ &= (1001001)_F \end{aligned}$$

We need extra rules to simplify the above expressions.

Playing a game with strings

The *Zeckendorf game* consists of moves

- $F_i + F_{i+1} \to F_{i+2}$
- $2F_i \to F_{i-2} + F_{i+1}$
- $2F_2 \to F_1 + F_3$
- $2F_1 \to F_2$

Theorem (BEFM '18)

Given any sum of Fibonacci numbers as an initial state:

1. The Zeckendorf game terminates.
2. The game ends in the Zeckendorf decomposition.

Structure of Addition

Recall $Z(X) :=$ number of summands in the Zeckendorf decomposition of X.

Observe that $Z(X + Y) \leq Z(X) + Z(Y)$.

- *Play the Zeckendorf game.*

What are other properties of the number of summands of Zeckendorf decompositions under addition?

Question

- Fix t.
- Pick random X represented by a string of size n.

	F_n	F_{n-1}	F_{n-2}	F_{n-3}	$\cdots$	F_2	F_1
$X =$	1	0	?	?	$\cdots$	?	?

Equivalent to $X \in [F_n, F_{n+1})$. There are F_{n-1} of such numbers. $(\star)$

How to compare

$$Z(X) \quad \text{and} \quad Z(X+t)?$$

Question

- Let t be a non-negative integer.
- Let X be a random integer uniformly chosen in $[F_n, F_{n+1})$.

What is the distribution of

$$\Delta_t Z(X) := Z(X+t) - Z(X)?$$

Result for $t = 1$

For $t = 1$, we can describe the distribution of $\Delta Z(X)$ exactly.

Theorem (SMALL '22)

Let X_n be a random variable uniformly chosen in $[F_n, F_{n+1})$. As $n \to \infty$,

ℓ	1	0	-1	-2	$\cdots$
$\mathbb{P}(\Delta Z(X_n) = \ell)$	ϕ^{-2}	ϕ^{-2}	ϕ^{-4}	ϕ^{-6}	$\cdots$

Proof for $t = 1$ and $\ell = 1$: $\mathbb{P}(\Delta Z(X) = 1) = \phi^{-2}$

$\Delta Z(X) = 1$ iff $Z(X+1) = Z(X) + 1$ iff

	F_n	F_{n-1}	F_{n-2}	F_{n-3}	$\cdots$	F_3	F_2	F_1
$X =$	1	0	?	?	$\cdots$	?	0	0

There are F_{n-3} out of F_{n-1} numbers satisfying this property. $(\star)$

Therefore,

$$\mathbb{P}(\Delta Z(X) = 1) = \frac{F_{n-3}}{F_{n-1}} \to \phi^{-2}.$$

Similar analysis works for $t = 1$ and arbitrary ℓ.

Result for arbitrary t

For arbitrary t, we can describe the distribution of $\Delta_t Z(X)$ for large negative values.

Theorem (SMALL '22)

Let k be a positive integer. Let $t \in [F_k, F_{k+1})$. Let $\ell < -k/2 = -O(\log t)$ be an integer. Let X_n be a random variable uniformly chosen in $[F_n, F_{n+1})$. As $n \to \infty$,

$$\frac{\mathbb{P}(\Delta_t Z(X_n) = \ell)}{\mathbb{P}(\Delta_t Z(X_n) = \ell + 1)} \to \phi^{-2}.$$

The limiting distribution of $\Delta_t Z(X_n)$ displays a geometric decay in the interval $(-\infty, -O(\log t))$.

Shifting to Random Shifts

For fixed t and random X, we can compare

$$Z(X+t) \qquad \text{vs.} \qquad Z(X).$$

For random X and Y, can we compare

$$Z(X+Y) \qquad \text{vs.} \qquad Z(X) \text{ and } Z(Y)?$$

Our Statistic

Recall that $Z(X+Y) \leq Z(X) + Z(Y)$.

We therefore define

$$T(X, Y) := Z(X) + Z(Y) - Z(X+Y)$$

Conjecture (SMALL '22)

Choosing X_n and Y_n uniformly from $[F_n, F_{n+1})$, the statistic $T(X_n, Y_n)$ is asymptotically normal as $n \to \infty$.

Why?

Gaussianity

Theorem (KKMW '10)

The random variable $Z(X_n)$ is asymptotically normal, in the sense that

$$\frac{Z(X_n) - \mathbb{E}[Z(X_n)]}{\sqrt{\mathrm{Var}[Z(X_n)]}} \xrightarrow{\text{dist}} \mathcal{N}(0,1)$$

Proof: *De Moivre–Laplace type argument. Sterling formula.*

$$\mathbb{P}(Z(X_n) = k) = \frac{1}{F_{n-1}}\binom{n-k}{k-1}$$

To generalize from Z to T, we need to think stochastically.

Lekkerkerker's Theorem

Theorem

Let $X_n \in [F_n, F_{n+1})$ be a uniformly distributed integer. The expectation of the number of summands $Z(X_n)$ is

$$\mathbb{E}[Z(X_n)] = \frac{n}{\phi^2 + 1} + O(1).$$

Our alternative approach (stochastic processes) is that

$$\mathbb{P}(Z_i(X_n) = 1) = \frac{F_{i-1}F_{n-i-2}}{F_{n-1}} = \frac{1}{\phi^2 + 1} + O(\phi^{-i})$$

Gaussianity Revisited

Theorem (KKMW '10)

$$\frac{Z(X_n) - \mathbb{E}[Z(X_n)]}{\sqrt{\mathrm{Var}[Z(X_n)]}} \xrightarrow{\text{dist}} \mathcal{N}(0,1)$$

Stochastic Proof:

- $Z_j(X_n) \sim \mathrm{Bern}(\frac{1}{\phi^2+1} + O(\phi^{-j}))$
- $\mathrm{Cov}(Z_r, Z_s) = O(\phi^{-|r-s|})$
- Apply CLT for strongly mixing r.v.'s

Going from Z to T

Before:

- $Z(X) = Z_1(X) + \cdots + Z_n(X)$

Now:

- $T(X, Y) = T_1(X, Y) + \cdots + T_{n+1}(X, Y)$
  - $T_j(X, Y) = Z_j(X) + Z_j(Y) - Z_j(X + Y)$

Method of Proof

We want to show $T(X_n, Y_n)$ is $\sim$ normal.

1. $T_j(X_n, Y_n)$ rapidly converges as $\lim_{j\to\infty} \lim_{n\to\infty} T_j(X_n, Y_n)$.
2. T_{j_1} and T_{j_2} are nearly uncorrelated for $|j_1 - j_2|$ large.
3. By CLT with strong mixing, we are done. □

Key Ingredient for Independence: Long runs

Gap Lemma

$$\mathbb{P}\left(\begin{array}{c} X, Y \text{ share a run} \\ \text{of 5 zeroes} \end{array}\right) = 1 - O(c^n)$$

for a constant $c \approx 0.93$.

	$\cdots$	F_{k+6}	F_{k+5}	F_{k+4}	F_{k+3}	F_{k+2}	F_{k+1}	F_k	$\cdots$
$X =$	$\cdots$	?	0	0	0	0	0	?	$\cdots$
$Y =$	$\cdots$	?	0	0	0	0	0	?	$\cdots$

Generalization

We believe normality of the statistic T also applies in the case of general *positive linear recurrence sequences*, such as

$$a_n := a_{n-1} + a_{n-2} + a_{n-3}$$

$$a_n := Ba_{n-1}$$

$$a_n := 2a_{n-1} + 3a_{n-2} + 20a_{n-17}$$

(all with canonical initial conditions)

References

Thank you!

- Paul Baird-Smith, Alyssa Epstein, Kristen Flint, and Steven J. Miller. "The Zeckendorf Game." In: *Combinatorial and additive number theory. III.* Vol. 297. Springer Proc. Math. Stat. Springer, Cham, 2020, pp. 25–28.
- Murat Koloğlu, Gene S. Kopp, Steven J. Miller, and Yinghui Wang. "On the number of summands in Zeckendorf decompositions." In: *Fibonacci Quart.* 49.2, 2011, pp. 116–130.