Continuing Analysis of the Zeckendorf Game

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Background

Zeckendorf Decomposition

$F_1 = 1, F_2 = 2, F_n = F_{n-1} + F_{n-2}$ are the Fibonacci numbers.

Theorem (Zeckendorf, 1972)

Every positive integer can be uniquely written as the sum of distinct non-adjacent Fibonacci numbers.

For example,

$$2022 = 1597 + 377 + 34 + 13 + 1 = F_{16} + F_{13} + F_8 + F_6 + F_1.$$

We denote this decomposition of 2022 by

$$F_{16} \wedge F_{13} \wedge F_8 \wedge F_6 \wedge F_1.$$

Computing the Zeckendorf Decomposition

Theorem (Zeckendorf, 1972)

Every $N \in \mathbb{Z}$ has a unique Zeckendorf decomposition: a decomposition into distinct non-adjacent Fibonacci numbers.

Given any decomposition of N, what makes it **not** the Zeckendorf decomposition? How can we make it "more Zeckendorf?"

$$\begin{aligned} F_{k-1} \wedge F_k &\mapsto F_{k+1} \\ F_k \wedge F_k &\mapsto F_{k-2} \wedge F_{k+1} \\ F_2 \wedge F_2 &\mapsto F_1 \wedge F_3 \\ F_1 \wedge F_1 &\mapsto F_2 \end{aligned}$$

Zeckendorf Game

In 2018, Baird-Smith, Epstein, Flint, and Miller introduced a two-player game based on this.

- Fix an integer N. The starting position is

$$F_1 \wedge \cdots \wedge F_1 = NF_1.$$

- Each turn, one of two kinds of moves can be chosen:

$$\begin{array}{rrrcl} \text{Combine:} & C_k: & F_{k-1} \wedge F_k & \mapsto & F_{k+1} \\ & C_1: & F_1 \wedge F_1 & \mapsto & F_2 \\ \text{Split:} & S_k: & F_k \wedge F_k & \mapsto & F_{k-2} \wedge F_{k+1} \\ & S_2: & F_2 \wedge F_2 & \mapsto & F_1 \wedge F_3 \end{array}$$

- Players alternate turns and the last player to move wins.

Another perspective on game states

We can interpret a game state as a distribution of tokens in a board of bins labeled with the Fibonacci numbers. For example, the initial game state is:

F_1	F_2	F_3	$\cdots$	F_n	$\cdots$
N	0	0	$\cdots$	0	$\cdots$

- The weighted sum remains constant equals N when applying moves.

Example of Game

$$
\begin{aligned}
\text{Combine:} \quad & C_k: & F_{k-1} \wedge F_k & \mapsto F_{k+1} & (k \geq 2) \\
& C_1: & F_1 \wedge F_1 & \mapsto F_2 & \\
\text{Split:} \quad & S_k: & F_k \wedge F_k & \mapsto F_{k-2} \wedge F_{k+1} & (k \geq 3) \\
& S_2: & F_2 \wedge F_2 & \mapsto F_1 \wedge F_3 &
\end{aligned}
$$

Properties of the Zeckendorf game

Proposition (Baird-Smith et al., 2018)

The game is guaranteed to terminate, and it terminates in the Zeckendorf decomposition.

The key idea is to assign a value to each configuration, which always decreases when applying moves.

Theorem (Baird-Smith et al., 2018)

For all $N > 2$, Player 2 has a winning strategy.

The proof of this theorem is non-constructive; no explicit winning strategy is known.

Game Lengths

Game Lengths

Prior works have established bounds on the lengths of games.

Theorem (Baird-Smith et al., 2018)

The minimum game length is $N - Z(N)$, where $Z(N)$ is the number of summands in the Zeckendorf decomposition

Theorem (Cusenza et at., 2022)

The upper bound for game length is

$$\phi^2 N - IZ(N) - \phi Z(N),$$

where $IZ(N)$ is the sum of the indices in the Zeckendorf decomposition.

For example, $IZ(2022) = 16 + 13 + 8 + 6 + 1 = 44$ and $Z(2022) = 5$.

Game Moves

Cusenza et at. defined MC_k and MS_k, for the number of times C_k and S_k are performed.

Lemma (SMALL 2022)

Let n be the largest summand in the Zeckendorf decomposition of N, we get that for any $2 \leq k \leq n-1$, the following sum is constant:

$$MS_k + MC_k + MC_{k+1} + \cdots + MC_{n-1}$$

We prove this by relabeling the board as follow:

F_1	$\cdots$	F_k	$F_{k+1}-1$	$F_{k+2}-2$	$F_{k+3}-4$	$F_{k+4}-7$	$\cdots$

where after the k^{th} bin, the value of a bin is equal to one less than the sum of the values of the two bins which precede it.

Game Moves

Lemma (SMALL 2022)

For any game with starting tokens N, we have that $MC_1 - MS_2 \approx (2-\phi)N$ with approximation error at most $\phi - 1$

We prove this with a relabeling of the board

2	3	5	$\cdots$	F_{k+1}	$\cdots$

and observing that the sum of token values goes from $2N$ to $\sim \phi N$ with the change coming only from C_1 or S_2 moves.

Proposition (SMALL 2022)

The upper bound for game length is

$$\phi^2 N - IZ(N) - 2Z(N) + (\phi - 1).$$

Possible Lengths

Existence of All Game Lengths

Prior work (Baird-Smith et al., 2018, and Cusenza et at., 2022):

- A shortest game: Play only combine moves. Guaranteed if play the rightmost possible combine move.
- A longest game: Play C_1 and splitting moves whenever possible; otherwise, play the leftmost combine moves.

Theorem (SMALL 2022)

Let M be the length of the longest game with input N. For any k such that $N - Z(N) \leq k \leq M$, there exists a Zeckendorf game of length k starting with input N.

Sketch for All Game Lengths

We establish this inductively on input $F_n \leq N < F_{n+1}$ to the game. The idea is to initially play "subgames" on smaller input.

$$\begin{aligned} & F_1 \wedge (N-1)F_1 \\ \xrightarrow{\text{all games}} \; & F_1 \wedge (\text{Z.d. of } N-1) \\ \xrightarrow{\text{unique game}} \; & (\text{Z.d. of } N) \end{aligned}$$

- These games yield an interval $[L_1, R_1]$ of game lengths.
- A shortest game is of this form.
  - Only combine moves $\implies$ shortest game.
  - The latter sequence of moves only uses combine moves.

Sketch for All Game Lengths

Similarly,

$$
\begin{aligned}
& (N - F_n + 1)F_1 \wedge (F_n - 1)F_1 \\
\xrightarrow{\text{all games}}\ & (N - F_n + 1)F_1 \wedge (\text{Z.d. of } F_n - 1) \\
\xrightarrow{\text{longest game}}\ & (\text{Z.d. of } N)
\end{aligned}
$$

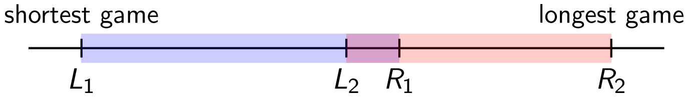
- These games yield an interval $[L_2, R_2]$ of game lengths.
- A longest game is of this form.
  - Play C_1 and S_k whenever possible $\implies$ longest game.
  - There is a game on $F_n - 1$ that only uses C_1 and S_k.

Sketch for All Game Lengths

The first family of games yields game lengths $[L_1, R_1]$;
The second family of games yields game lengths $[L_2, R_2]$.

We check the following items:

- A shortest game is in the first family.
- A longest game is in the second family.
- $L_2 \leq R_1$.



Thus, $[L_1, R_1] \cup [L_2, R_2]$ is the interval from the shortest game length to the longest game length.

Random Games

Equal Probability of Winning Randomized Game

Baird-Smith et al. introduced the notion of a random Zeckendorf game.

Definition (Baird-Smith et al., 2018)

A **random Zeckendorf game** is one such that on every turn, a player picks from all legal moves with equal probability.

This suggests the study of probabilistic aspects of random Zeckendorf games. In particular, we have the following motivating conjecture in this direction.

Conjecture (Baird-Smith et al., 2018)

As $N \to \infty$, the number of moves in a random game where all legal moves are equally likely converges in distribution to a Gaussian.

Gaussianity Conjecture: Empirics

Baird-Smith et al. verify this result empirically, executing 9999 random Zeckendorf games for $N = 200$, and plotting the empirical distribution against a Gaussian.

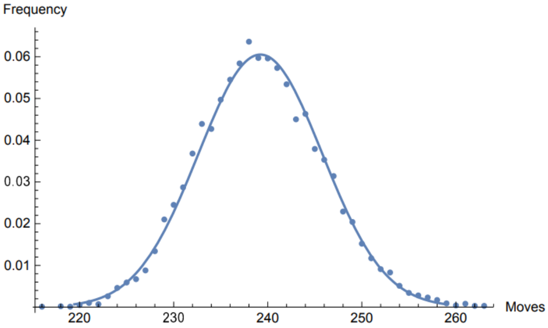


Figure: Result of running 9999 simulations of random Zeckendorf games on input $N = 200$, plotted against a Gaussian.

Winning Odds

In ongoing work towards resolving this conjecture, we have the following initial result.

Theorem (SMALL 2022)

As $N \to \infty$, the probability that each player wins in a random game approaches $1/2$. Explicitly,

$$\lim_{N\to\infty} \mathbb{P}_N(\textit{Player 1 wins}) = \lim_{N\to\infty} \mathbb{P}_N(\textit{Player 2 wins}) = \frac{1}{2}$$

where $\mathbb{P}_N$ denotes the probability measure induced by playing random Zeckendorf games with input N.

Sketch for Winning Odds

- For $k \geq 2$, if bin $k-1$ has height at least 1 and bin k has height at least 2, the move C_k and sequence (S_k, C_{k-1}) are both legal and have the same effect on the game.
- Partition the collection of all Zeckendorf games on input N into those with the same **base sequence**, obtained from replacing certain instances of (S_k, C_{k-1}) by C_k. Let us index these sets by $\{\mathcal{A}_i : i \in \mathcal{I}\}$.
- By the law of total probability,

$$\mathbb{P}_N(\text{P1 wins}) = \sum_{i \in \mathcal{I}} \mathbb{P}_N\left(\text{P1 wins} \mid \mathcal{A}_i\right) \cdot \mathbb{P}_N\left(\mathcal{A}_i\right).$$

- The event P1 wins, conditioned on the event $\mathcal{A}_i$, can be understood as a fixed constant added to a binomial random variable with exploding variance, which converges to being odd roughly half the time.

Future Works

- Improve the upper bound of the game length.
- Prove Gaussianity of the game length of random games.
- Find explicit winning strategy for Player 2.
    - Maybe for specific values of N.

Acknowledgments and References

Thank you! Questions?

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