Variants on the Zeckendorf Game

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Supported by Dr. Herchel Smith Fellowship Fund, Emmanuel College University of Cambridge, NSF, Williams College, Finnerty Fund

Affiliated with Williams College, University of Cambridge, University of Rhode Island

Young Mathematicians Conference
July 30, 2026

Williams SMALL
Probability and Number Theory

Introduction and Definitions

Zeckendorf Decomposition:

- Zeckendorf's Theorem: Every integer n can be written as a unique sum of non-adjacent Fibonacci Numbers.
- For example:

$$\begin{aligned} 10 &= 2 + 8 \\ 27 &= 1 + 5 + 21 \\ 66 &= 5 + 8 + 55 \end{aligned}$$

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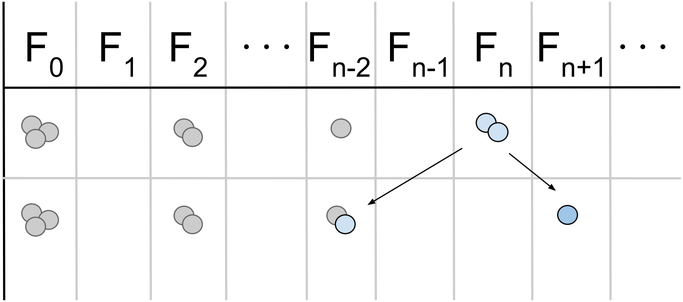
- The game terminates when the full Zeckendorf decomposition is reached.

Zeckendorf Game Example Move



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Zeckendorf Game Example: $n = 6$

n=6	1	2	3	5	8
	6				
P1	4	1			
P2	3	0	1		
P1	1	1	1		
P2	1	0	0	1	

- Baird-Smith, Epstein, Flint, Miller proved nonconstructively that for $n > 2$, Player 2 has the winning strategy.

Highest Summand Zeckendorf Game

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Who wins?

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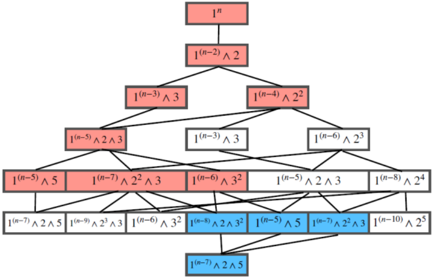
Original Game: Player 2 always wins

- Non-Constructive Proof
- Relies on Player 2 “stealing" Player 1’s strategy.



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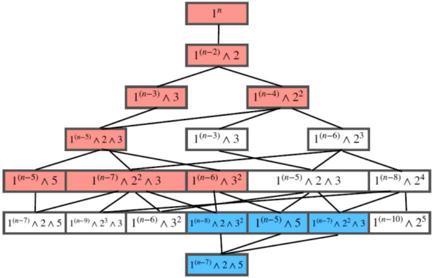

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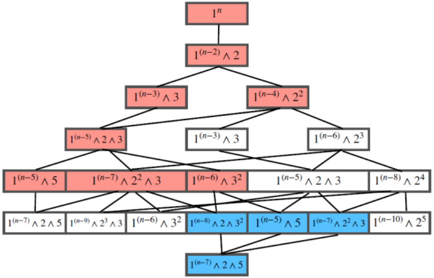

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Proof applies to many variants where winning is conditional on certain game state.

Zeckendorf Game Invariants

- Let the box i have $a_i^{(k)}$ coins after k moves of the Zeckendorf game
- Define a polynomial

$$p_1^{(k)}(x) = \sum_{i=1}^{T} a_i^{(k)} x^i$$

- Let $\psi = \frac{1+\sqrt{5}}{2}, \varphi = \frac{1-\sqrt{5}}{2}$ be the two roots of $x^2 - x - 1 = 0$.
- The legal moves of the Zeckendorf game are divided into
  - Combining moves $\begin{cases} C_1: & 2F_1 \longrightarrow F_2 \\ C_i: & F_{i-1} + F_i \longrightarrow F_{i+1} \quad i \geq 2 \end{cases}$
  - Splitting moves $\begin{cases} S_2: & 2F_2 \longrightarrow F_1 + F_3 \\ S_j: & 2F_j \longrightarrow F_{j-2} + F_{j+1} \quad j \geq 3 \end{cases}$
- For a completed game, let MC_i denote the total number of moves C_i, and let MS_j denote the total number of moves S_j.

Zeckendorf Game Invariants

Lemma 1

For a Zeckendorf game of n, define $p_1^{\text{start}}(x) = nx$ and let $p_1^{\text{final}}(x)$ be the polynomial corresponding to the final Zeckendorf decomposition. Then

$$\frac{p_1^{\text{final}}(\phi) - p_1^{\text{start}}(\phi)}{\varphi} = MC_1 - MS_2.$$

Proof.

- Evaluate $p_1^{(k+1)}(x) - p_1^{(k)}(x)$ at the positive fibonacci root ϕ
- Due to the root satisfying the recursive equation $\phi^2 = \phi + 1$, obtain that the polynomial changes only at the C_1 and S_2 moves
- Hence deduce that $MC_1 - MS_2$ is fixed as the inital and final polynomal are fixed

Zeckendorf Game Invariants

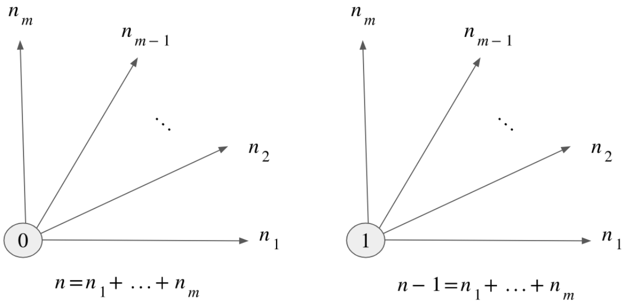
- For $2 \leq t \leq T-1$, by considering similar polynomials $p_t^{(k)}(x) = \sum_{i=t}^{T} a_i^{(k)} x^{i-t}$, obtain that $MC_i + MS_i - MS_{i+1}$ are fixed for all i
- As the values are fixed, and we know that the shortest C-vector is unique, then we obtain the following corollary:

$$MC_i + MS_i - MS_{i+1} = MC_i^{min}$$

- Note that C-vector uniquely defines the set of moves used in the game, both combine and split, due to the equations

m-branch Zeckendorf Game

- Let the box i on branch b have $a_{b,i}^{(k)}$ after k moves of the m-branch Zeckendorf game
- The terminal state determines a tuple $(n_1, \ldots, n_m)$.
- For each branch b, define $n_b = \sum_{i=2}^{T} a_{b,i}^{\text{final}} F_i$ where $a_{b,i}^{\text{final}}$ denotes the terminal number of coins in box i of branch b.



m-branch Zeckendorf Game

Theorem 2

Consider an m-branch Zeckendorf game of n that terminates with a fixed tuple

$$(n_1, \ldots, n_m).$$

Then, for every $1 \leq i \leq T-1$, the quantity

$$\boxed{MC_i + MS_i - MS_{i+1}}$$

is the same for every game terminating with this tuple. Here we use the convention

$$MS_1 = 0.$$

m-branch Zeckendorf Game

Corollary 3

Suppose an m-branch Zeckendorf game of n terminates with the tuple

$$(n_1, \dots, n_m).$$

Then, for every $1 \le i \le T-1$,

$$\boxed{MC_i + MS_i - MS_{i+1} = \sum_{b=1}^{m} MC_i^{\min}(n_b),}$$

provided the comparison game on each branch is chosen to be a shortest one-branch Zeckendorf game for n_b.

Player 2 Winning Condition

- Recall the equations we previously obtained

$$MC_i + MS_i - MS_{i+1} = MC_i^{min}$$

- As a consequence, after appropriate addition, we obtain another invariant in the game:

$$h = \sum_{i=1}^{N-1} iMC_i + MS_i$$

Player 2 Winning Condition

Corollary 4

Let

$$E = \sum_{\substack{1 \le i \le r \\ i \text{ even}}} MC_i$$

be the total number of even-indexed combining moves in a complete game. Then

$$L \equiv h + E \bmod 2$$

Consequently, Player 2 makes the final move if and only if

$$E \equiv h \bmod 2$$

.

Proof of Winning Condition

Proof: The length of the game is

$$L = \sum_{i=1}^{N} MC_i + \sum_{j=2}^{N} MS_j$$

By the definition of h,

$$h = \sum_{i=1}^{N} iMC_i + \sum_{j=2}^{N} MS_j$$

Subtracting gives

$$h - L = \sum_{i=1}^{N} (i-1)MC_i$$

Player 2 makes the final move exactly when the game length L is even. Hence, player 2 wins if and only if

$$h + E \equiv 0 \bmod 2$$

Theorem about the Longest Game

Theorem 5

The longest Zeckendorf game has a unique set of moves.

Proof.

For any longest game not in the LA family, find the two games in the LA family and the new family respectively s.t. the matching sequence of moves k is maximized. Now can show that if both are the longest k is not maximal unless k is the longest length of the game. Therefore obtain that the longest Zeckendorf game has a unique family of moves. □

Theorem about the Longest Game

Corollary

The length of the longest game can be computed using the maximal and minimal combining vectors as follows:

$$L^{\max} - L^{\min} = \sum_{i=1}^{N-1} (N-1-i) \left(MC_i^{\max} - MC_i^{\min} \right).$$

Future Work

- Use the observations for finding the constructive strategy for Player 2
- Explicit derivation of the longest C-vector, which would give us the precise bound on the longest game
- Finding proof for Player 2 winning the 2-branch game

References

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Thank You!

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