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Zeckendorf-Niven Numbers in Arithmetic Progressions

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Fibonacci Numbers

Fibonacci Numbers

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

$$\begin{aligned} 1 + 2 &= 3 & 8 + 13 &= 21 \\ 2 + 3 &= 5 & 13 + 21 &= 34 \\ 3 + 5 &= 8 & 21 + 34 &= 55 \\ 5 + 8 &= 13 & 34 + 55 &= 89 \end{aligned}$$

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Zeckendorf's Theorem

Zeckendorf's Theorem

Every positive integer, n, can be uniquely written as the sum of distinct and non-consecutive Fibonacci numbers.

Examples:

$$34 = 34, \qquad 12 = 8 + 3 + 1, \qquad 63 = 55 + 8$$

Zeckendorf-Niven Number

Definition

We call a positive integer n *Zeckendorf-Niven* if the number of terms in its Zeckendorf decomposition (denoted by $S_Z(n)$) divides n.

Examples:

4 = 3 + 1 is Zeckendorf-Niven

7 = 5 + 2 is NOT Zeckendorf-Niven

Focus of Research

For any d, what's the longest sequence i.e.,

$$n, n+d, n+2d, n+3d, \ldots$$

where all the terms are Zeckendorf-Niven.

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Arithmetic Progressions of ZN Numbers

Theorem (Grundman 2007)

- Any sequence of 5 or more consecutive Zeckendorf-Niven numbers are a subsequence of 1,2,3,4,5,6.

Theorem

- Any sequence of 8 or more Zeckendorf-Niven numbers in 2-AP are a subsequence of 2, 4, 6, 8, 10, 12, 14, 16, 18.

Arithmetic Progressions of ZN Numbers

Theorem

For any Fibonacci $F_a \geq 2$, there exists an infinite number of Zeckendorf-Niven sequences $n, n + F_a, n + 2F_a$ such that $S_z(n) = S_z(n + F_a) = S_z(n + 2F_a) = y$ where $y \geq 2$ is a factor of F_a.

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Overview of Proof

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Proof Strategy

For $F_6 = 8$ and $y = 2$, consider

$$5, 13, 21$$

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Notice that $5 \equiv 13 \equiv 21 \pmod{2}$.

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Find a Fibonacci $F \equiv 5 \equiv 13 \equiv 21 \pmod{2}$.

$$F + 5 = F + 5, \quad (F + 5) + 8 = F + 13, \quad (F + 5) + 2 \cdot 8 = F + 21$$

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Examples

Example

For $F_6 = 8$ and $y = 2$,

$$60 = 55 + 5, \quad 68 = 55 + 13, \quad 76 = 55 + 21$$

For $F_6 = 8$ and $y = 4$,

$$1924 = 1597 + 233 + 89 + 5$$
$$1932 = 1597 + 233 + 89 + 13$$
$$1940 = 1597 + 233 + 89 + 21$$

Infinitely Many ZN Numbers

Theorem

Every arithmetic progression

$$n, n + d, n + 2d, \ldots$$

has infinitely many Zeckendorf-Niven numbers.

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Thank you!

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