

Zeckendorf-Niven Numbers in Arithmetic Progressions

Kelly Lao, Steven Miller, Nick Rosa*, Mark Shiliaev*, Garrett Tresch, Wing Hong Tony Wong, Han Zhang

kellylao23@gmail.com, sjm1@williams.edu, nicholasrosa123@gmail.com,
mshiliaev@tamu.edu, treschgd@tamu.edu, wong@kutztown.edu,
hzhang23@ole.augie.edu,

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Overview

- 1 Introduction
- 2 ZN Numbers in 2-APs
- 3 Results in F_a -APs
- 4 ZN Numbers in every AP
- 5 Future Work / Refs

Niven Numbers

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We call a natural number $n \in \mathbb{N}$ a *Niven* number if it is divisible by the sum of its digits.

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Example:

135 is *Niven* as $9 \mid 135$

89 is not *Niven* as $17 \nmid 89$

Fibonacci Numbers

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad F_4 = 3, \quad F_5 = 5, \quad F_6 = 8, \quad F_7 = 13, \\ F_8 = 21, \quad F_9 = 34, \quad \dots$$

$$F_n = F_{n-1} + F_{n-2}$$

Zeckendorf's Theorem

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Example:

$$34 = 34, \quad 12 = 8 + 3 + 1, \quad 63 = 55 + 8$$

Zeckendorf-Niven number

Definition

We call a positive integer n *Zeckendorf-Niven* if the number of terms in its Zeckendorf decomposition (denoted by $S_z(n)$) divides n .

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Example:

$$159 = 144 + 13 + 2 \text{ is } \textit{Zeckendorf-Niven}$$

$$7 = 5 + 2 \text{ is not } \textit{Zeckendorf-Niven}$$

Focus of Research

For some $d \in \mathbb{N}$, what's the maximum length of a sequence i.e.,

$$n, n + d, n + 2d, n + 3d, \dots$$

where all the terms are Zeckendorf-Niven.

Previous Work

Theorem (Grundman 2007)

- Any sequence of 5 or more consecutive Zeckendorf-Niven numbers are a subsequence of 1,2,3,4,5,6.
- There are infinitely many sequences of 4 consecutive Zeckendorf-Niven numbers.

Zeckendorf-Niven Numbers in 2-APs

Theorem (Grundman 2007)

- Any sequence of 5 or more consecutive Zeckendorf-Niven numbers are a subsequence of 1,2,3,4,5,6.
- There are infinitely many sequences of 4 consecutive Zeckendorf-Niven numbers.

Theorem

- Any sequence of 8 or more Zeckendorf-Niven numbers in 2-AP are a subsequence of 2, 4, 6, 8, 10, 12, 14, 16, 18.
- There are infinitely many 2-AP of 5 Zeckendorf-Niven numbers.

Adding a Restriction

For some $d \in \mathbb{N}$, what's the maximum length of a sequence i.e.,

$$n, n + d, n + 2d, n + 3d, \dots$$

where $S_z(n) = S_z(n + d) = \dots$ and all the terms are Zeckendorf-Niven.

Results in F_a -APs

Theorem

For any Fibonacci $F_a \geq 2$, there exists an infinite number of Zeckendorf-Niven sequences $n, n + F_a, n + 2F_a$ such that $S_z(n) = S_z(n + F_a) = S_z(n + 2F_a) = y$ where $y \geq 2$ is a factor of F_a . No sequence of length 4 exists under similar conditions.

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Example for $F_a = 3$ and $y = 3$:

$$324 = 233 + 89 + 2, \quad 327 = 233 + 89 + 5, \quad 330 = 233 + 89 + 8$$

Zeckendorf-Niven numbers in every AP

Pisano Period Lemma

For every $n \in \mathbb{N}$ the sequence $(F_k)_{k=0}^{\infty}$ is periodic modulo n .

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The period of sequence $(F_k)_{k=0}^{\infty}$ modulo n is called the n th *Pisano period* and is denoted by $\pi(n)$.

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Example: For $n = 4$:

$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 1, F_6 = 0, F_7 = 1,$
so $\pi(4) = 6$.

Zeckendorf-Niven numbers in every AP

Lemma

For every $n, m \in \mathbb{N}$, $n \mid F_{m\pi(n)}$.

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For every $b, m, d \in \mathbb{N}$, there exists $n \in \mathbb{N}$, such that $S_z(b) + m = S_z(nd + b)$.

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The main idea of the result:

$$N = \sum_{i=1}^m F_{a_i} \rightarrow F_{a_m + \pi(d)} + \sum_{i=1}^{m-1} F_{a_i} \rightarrow F_{a_m + k\pi(d)} + \sum_{i=1}^{m-1} F_{a_i} = N',$$

$$\gcd(\pi(m), \pi(d)) = 1.$$

Zeckendorf-Niven numbers in every AP

Lemma (Koshy 2001)

For all $n \in \mathbb{N}$ we have $\pi(F_{2n}) = 4n$ and $\pi(F_{2n+1}) = 8n + 4$.

Zeckendorf-Niven numbers in every AP

Lemma (Koshy 2001)

For all $n \in \mathbb{N}$ we have $\pi(F_{2n}) = 4n$ and $\pi(F_{2n+1}) = 8n + 4$.

Theorem

For every $d, b \in \mathbb{N}$ there is infinitely many Zeckendorf-Niven numbers in the sequence $\{nd + b\}_{n=0}^{\infty}$.

Generalization To The Lucas Sequence

Lucas sequence: $L_{n+1} = L_n + L_{n-1}$;

$$L_0 = 2, L_1 = 1, L_2 = 3, L_3 = 4, \dots$$

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$$L_0 = 2, L_1 = 1, L_2 = 3, L_3 = 4, \dots$$

Theorem

For every $d, b \in \mathbb{N}$ there is infinitely many Lucas-Niven numbers in the sequence $\{nd + b\}_{n=0}^{\infty}$.

Future Work

- Generalize the previous theorem for any decomposition based on an positive linear recurrence sequence (PLRS).

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}$$

- Generalize theorems on consecutive numbers with a non-Fibonacci difference and with the same number of coefficients in their Zeckendorf decompositions.
- Generalize the bounds for the number of consecutive Zeckendorf-Niven numbers in an arbitrary arithmetic progression.

Thank You!

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