The Theory of Normalization Constants and Zeckendorf Decompositions

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Introduction

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$.

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Example: $51 = ?$

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Example: $51 = 34 + 17 = F_8 + 17$.

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Example: $51 = 34 + 13 + 4 = F_8 + F_6 + 4$.

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Example: $51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + F_1$.
Example: $83 = 55 + 21 + 5 + 2 = F_9 + F_7 + F_4 + F_2$.
Observe: 51 miles $\approx$ 82.1 kilometers.
Introducing: kmarathon (16.2799), maraphion (16.1925), kmaraphion (10.0615)....

Old Results: $f(x) = \exp(-(x-\mu)^2/2\sigma^2)/\sqrt{2\pi\sigma}$

Central Limit Type Theorem

As $n \to \infty$ distribution of number of summands in Zeckendorf decomposition for $m \in [F_n, F_{n+1})$ is Gaussian (normal).

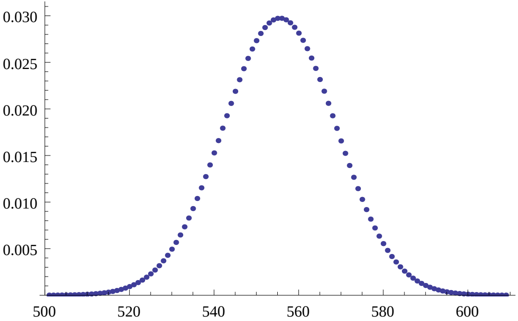

Figure: Number of summands in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

SMALL REU Results: Bulk Gaps: $m \in [F_n, F_{n+1})$ and $\phi = \frac{1+\sqrt{5}}{2}$

$$m = \sum_{j=1}^{k(m)=n} F_{i_j}, \quad \nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right).$$

Theorem (Zeckendorf Gap Distribution)

Gap measures $\nu_{m;n}$ converge almost surely to average gap measure where $P(k) = 1/\phi^k$ for $k \geq 2$.

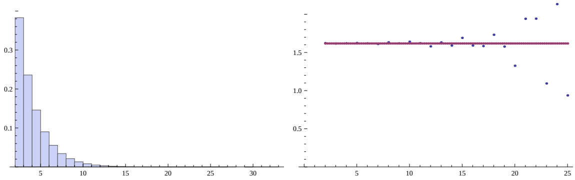

Figure: Distribution of gaps in $[F_{1000}, F_{1001})$; $F_{2010} \approx 10^{208}$.

SMALL REU : Longest Gap

Theorem (Longest Gap)

As $n \to \infty$, the probability that $m \in [F_n, F_{n+1})$ has longest gap less than or equal to $f(n)$ converges to

$$\mathrm{Prob}\left(L_n(m) \leq f(n)\right) \;\approx\; e^{-e^{\log n - f(n)/\log \phi}}.$$

Immediate Corollary: If $f(n)$ grows ***slower*** or ***faster*** than $\log n/\log \phi$, then $\mathrm{Prob}(L_n(m) \leq f(n))$ goes to **0** or ***1***, respectively.

Preliminaries: The Cookie Problem

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Divides the cookies into P sets.

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Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.

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Let $p_{n,k} = \#\left\{N \in [F_n, F_{n+1})\text{: the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands}\right\}$.

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For $N \in [F_n, F_{n+1})$, the largest summand is F_n.

$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$
$$1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n, \quad i_j - i_{j-1} \geq 2.$$

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$$d_1 := i_1 - 1, \quad d_j := i_j - i_{j-1} - 2 \ (j > 1).$$
$$d_1 + d_2 + \cdots + d_k = n - 2k + 1, \quad d_j \geq 0.$$

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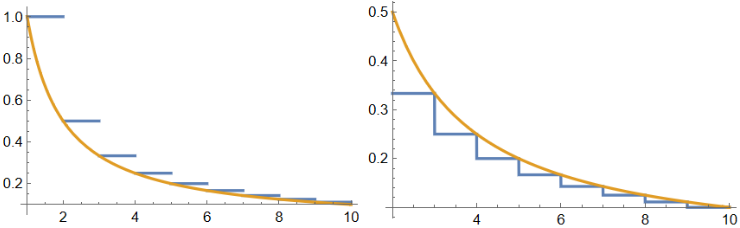
Cookie counting $\Rightarrow$ $p_{n,k} = \binom{n-2k+1\ +\ k-1}{k-1} = \binom{n-k}{k-1}$.

Gaussian Behavior

Integral Test: Theory

Integral Test: If a_n monotonic and similar f with $f(n) = a_n$ then $a_1 + \cdots + a_n \approx \int_1^n f(x)dx$.

Example: $a_n = 1/n$ so $1 + 1/2 + \cdots + 1/n \approx \int_1^n \frac{dx}{x} \ = \ \log n$.

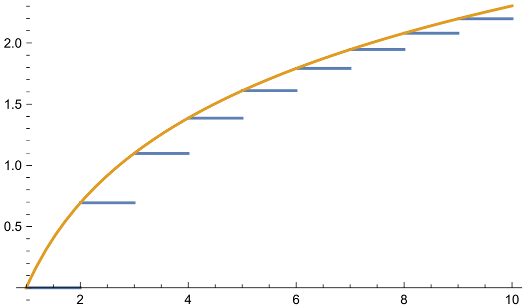


Integral Test: Stirling's Formula: Study $\log(n!) = \log 1 + \cdots + \log n$

Stirling's Formula: $n! \approx n^n e^{-n} \sqrt{2\pi n}$.

$$\log 1 + \cdots + \log n \approx \int_1^n \log x dx \;=\; [x \log x - x]_1^n \;=\; (n \log n - n) - 1$$

$$\text{Thus } n! \;=\; e^{n \log n - n - 1} \;=\; n^n e^{-n} \frac{1}{e}.$$



Preliminaries: Big-Oh Notation, Logarithms

- $f(x) = O(g(x))$ if for all x sufficiently large, there is a C with $|f(x)| \leq Cg(x)$.
  - ◇ $x = O(x^2)$, $\log x = O(x^r)$ for any $r > 0$.
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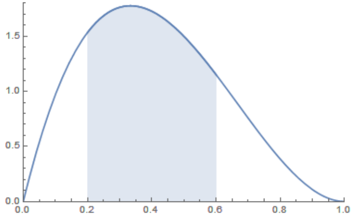
- $\log(xy) \ = \ \log x + \log y$.

- $\log(1+x) \ = \ x - \frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \cdots$ for $|x| < 1$.
  - ◇ $\log(1+x) \ = \ x + O(x^2)$
  - ◇ $\log(1+x) \ = \ x - \frac{x^2}{2} + O(x^3)$.

Preliminaries: Probability Review



- Let X be random variable with density $p(x)$:
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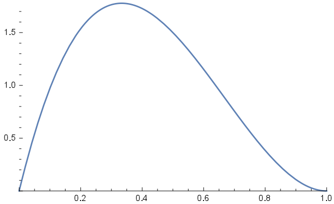

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- Mean $\mu = \int_{-\infty}^{\infty} xp(x)dx$.

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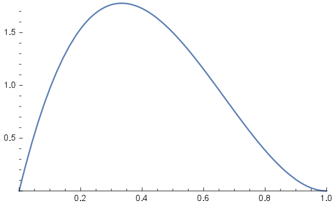

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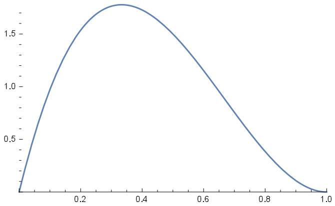

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- Mean $\mu = \int_{-\infty}^{\infty} xp(x)dx$.
- Variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)dx$.
- Independence: knowledge of one random variable gives no knowledge of the other.

Binet's Formula

Golden mean/ratio: $\phi = \frac{1+\sqrt{5}}{2}$.

If $F_0 = 0$ and $F_1 = 1$ then

$$F_n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}(1-\phi)^n.$$

If $F_1 = 1$ and $F_2 = 2$ then

$$F_n = \frac{\phi}{\sqrt{5}}\phi^n - \frac{1-\phi}{\sqrt{5}}(1-\phi)^n.$$

Computing Quickly: https://youtu.be/KzT9I1d-LlQ (Sheafification of G).

Generating Function (Example: Binet's Formula)

Binet's Formula

$$F_1 = F_2 = 1; \ F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

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&\Rightarrow g(x) - F_1 x - F_2 x^2 = x(g(x) - F_1 x) + x^2 g(x) \\
&\Rightarrow g(x) = x/(1 - x - x^2).
\end{aligned}
$$

Partial Fraction Expansion (Example: Binet's Formula)

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Coefficient of x^n (power series expansion):

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right]$$ - Binet's Formula!

(using geometric series: $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$).

Gaussian Behavior

Generalizing Lekkerkerker: Erdos-Kac type result

Presented CANT in May 2010, students started in June.

Theorem (KKMW 2010)

As $n \to \infty$, the distribution of the number of summands in Zeckendorf's Theorem is a Gaussian: $f(x) = \exp(-(x-\mu)^2/2\sigma^2)/\sqrt{2\pi\sigma}$.

Sketch of proof: Use Stirling's formula

$$n! \approx n^n e^{-n}\sqrt{2\pi n}$$

to approximates binomial coefficients, after a few pages of algebra find the probabilities are approximately Gaussian.

(Sketch of the) Proof of Gaussianity

The probability density for the number of Fibonacci numbers that add up to an integer in $[F_n, F_{n+1})$ is $f_n(k) = \binom{n-1-k}{k}/F_{n-1}$. Consider the density for the $n+1$ case. Then we have, by Stirling

$$\begin{aligned} f_{n+1}(k) &= \binom{n-k}{k}\frac{1}{F_n} \\ &= \frac{(n-k)!}{(n-2k)!k!}\frac{1}{F_n} = \frac{1}{\sqrt{2\pi}}\frac{(n-k)^{n-k+\frac{1}{2}}}{k^{(k+\frac{1}{2})}(n-2k)^{n-2k+\frac{1}{2}}}\frac{1}{F_n} \end{aligned}$$

plus a lower order correction term.
Also we can write $F_n = \frac{1}{\sqrt{5}}\phi^{n+1} = \frac{\phi}{\sqrt{5}}\phi^n$ for large n, where ϕ is the golden ratio (we are using relabeled Fibonacci numbers where $1 = F_1$ occurs once to help dealing with uniqueness and $F_2 = 2$). We can now split the terms that exponentially depend on n.

$$f_{n+1}(k) = \left(\frac{1}{\sqrt{2\pi}}\sqrt{\frac{(n-k)}{k(n-2k)}}\frac{\sqrt{5}}{\phi}\right)\left(\phi^{-n}\frac{(n-k)^{n-k}}{k^k(n-2k)^{n-2k}}\right).$$

Define

$$N_n = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{(n-k)}{k(n-2k)}}\frac{\sqrt{5}}{\phi}, \quad S_n = \phi^{-n}\frac{(n-k)^{n-k}}{k^k(n-2k)^{n-2k}}.$$

Thus, write the density function as

$$f_{n+1}(k) = N_n S_n$$

where N_n is the first term that is of order $n^{-1/2}$ and S_n is the second term with exponential dependence on n.

(Sketch of the) Proof of Gaussianity

Model the distribution as centered around the mean by the change of variable $k = \mu + x\sigma$ where μ and σ are the mean and the standard deviation, and depend on n. The discrete weights of $f_n(k)$ will become continuous. This requires us to use the change of variable formula to compensate for the change of scales:

$$f_n(k)dk = f_n(\mu + \sigma x)\sigma dx.$$

Using the change of variable, we can write N_n as

$$\begin{aligned} N_n &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n-k}{k(n-2k)}}\frac{\phi}{\sqrt{5}} \\ &= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-k/n}{(k/n)(1-2k/n)}}\frac{\sqrt{5}}{\phi} \\ &= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-(\mu+\sigma x)/n}{((\mu+\sigma x)/n)(1-2(\mu+\sigma x)/n)}}\frac{\sqrt{5}}{\phi} \\ &= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-C-y}{(C+y)(1-2C-2y)}}\frac{\sqrt{5}}{\phi} \end{aligned}$$

where $C = \mu/n \approx 1/(\phi+2)$ (note that $\phi^2 = \phi + 1$) and $y = \sigma x/n$. But for large n, the y term vanishes since $\sigma \sim \sqrt{n}$ and thus $y \sim n^{-1/2}$. Thus

$$N_n \approx \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-C}{C(1-2C)}}\frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{(\phi+1)(\phi+2)}{\phi}}\frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{5(\phi+2)}{\phi}} = \frac{1}{\sqrt{2\pi\sigma^2}}$$

since $\sigma^2 = n\frac{\phi}{5(\phi+2)}$.

(Sketch of the) Proof of Gaussianity

For the second term S_n, take the logarithm and once again change variables by $k = \mu + x\sigma$,

$$\begin{aligned}
\log(S_n) &= \log\left(\phi^{-n}\frac{(n-k)^{(n-k)}}{k^k(n-2k)^{(n-2k)}}\right)\\
&= -n\log(\phi) + (n-k)\log(n-k) - (k)\log(k)\\
&\quad - (n-2k)\log(n-2k)\\
&= -n\log(\phi) + (n-(\mu+x\sigma))\log(n-(\mu+x\sigma))\\
&\quad - (\mu+x\sigma)\log(\mu+x\sigma)\\
&\quad - (n-2(\mu+x\sigma))\log(n-2(\mu+x\sigma))\\
&= -n\log(\phi)\\
&\quad + (n-(\mu+x\sigma))\left(\log(n-\mu)+\log\left(1-\frac{x\sigma}{n-\mu}\right)\right)\\
&\quad - (\mu+x\sigma)\left(\log(\mu)+\log\left(1+\frac{x\sigma}{\mu}\right)\right)\\
&\quad - (n-2(\mu+x\sigma))\left(\log(n-2\mu)+\log\left(1-\frac{x\sigma}{n-2\mu}\right)\right)\\
&= -n\log(\phi)\\
&\quad + (n-(\mu+x\sigma))\left(\log\left(\frac{n}{\mu}-1\right)+\log\left(1-\frac{x\sigma}{n-\mu}\right)\right)\\
&\quad - (\mu+x\sigma)\log\left(1+\frac{x\sigma}{\mu}\right)\\
&\quad - (n-2(\mu+x\sigma))\left(\log\left(\frac{n}{\mu}-2\right)+\log\left(1-\frac{x\sigma}{n-2\mu}\right)\right).
\end{aligned}$$

(Sketch of the) Proof of Gaussianity

Note that, since $n/\mu = \phi + 2$ for large n, the constant terms vanish. We have $\log(S_n)$

$$
\begin{aligned}
= \quad & -n\log(\phi) + (n-k)\log\left(\frac{n}{\mu} - 1\right) - (n-2k)\log\left(\frac{n}{\mu} - 2\right) + (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-2\mu}\right) \\
= \quad & -n\log(\phi) + (n-k)\log\left(\phi+1\right) - (n-2k)\log\left(\phi\right) + (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-2\mu}\right) \\
= \quad & n(-\log(\phi) + \log\left(\phi^2\right) - \log\left(\phi\right)) + k(\log(\phi^2) + 2\log(\phi)) + (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu+x\sigma))\log\left(1 - 2\frac{x\sigma}{n-2\mu}\right) \\
= \quad & (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) \\
& - (n - 2(\mu+x\sigma))\log\left(1 - 2\frac{x\sigma}{n-2\mu}\right).
\end{aligned}
$$

(Sketch of the) Proof of Gaussianity

Finally, we expand the logarithms and collect powers of $x\sigma/n$.

$$\begin{aligned}
\log(S_n) &= (n-(\mu+x\sigma))\left(-\frac{x\sigma}{n-\mu}-\frac{1}{2}\left(\frac{x\sigma}{n-\mu}\right)^2+\dots\right)\\
&\quad -(\mu+x\sigma)\left(\frac{x\sigma}{\mu}-\frac{1}{2}\left(\frac{x\sigma}{\mu}\right)^2+\dots\right)\\
&\quad -(n-2(\mu+x\sigma))\left(-2\frac{x\sigma}{n-2\mu}-\frac{1}{2}\left(2\frac{x\sigma}{n-2\mu}\right)^2+\dots\right)\\
&= (n-(\mu+x\sigma))\left(-\frac{x\sigma}{n\frac{(\phi+1)}{(\phi+2)}}-\frac{1}{2}\left(\frac{x\sigma}{n\frac{(\phi+1)}{(\phi+2)}}\right)^2+\dots\right)\\
&\quad -(\mu+x\sigma)\left(\frac{x\sigma}{\frac{n}{\phi+2}}-\frac{1}{2}\left(\frac{x\sigma}{\frac{n}{\phi+2}}\right)^2+\dots\right)\\
&\quad -(n-2(\mu+x\sigma))\left(-\frac{2x\sigma}{n\frac{\phi}{\phi+2}}-\frac{1}{2}\left(\frac{2x\sigma}{n\frac{\phi}{\phi+2}}\right)^2+\dots\right)\\
&= \frac{x\sigma}{n}n\left(-\left(1-\frac{1}{\phi+2}\right)\frac{(\phi+2)}{(\phi+1)}-1+2\left(1-\frac{2}{\phi+2}\right)\frac{\phi+2}{\phi}\right)\\
&\quad -\frac{1}{2}\left(\frac{x\sigma}{n}\right)^2 n\left(-2\frac{\phi+2}{\phi+1}+\frac{\phi+2}{\phi+1}+2(\phi+2)-(\phi+2)+4\frac{\phi+2}{\phi}\right)\\
&\quad +O\left(n\,(x\sigma/n)^3\right)
\end{aligned}$$

(Sketch of the) Proof of Gaussianity

$$
\begin{aligned}
\log(S_n) &= \frac{x\sigma}{n} n \left(-\frac{\phi+1}{\phi+2}\frac{\phi+2}{\phi+1} - 1 + 2\frac{\phi}{\phi+2}\frac{\phi+2}{\phi} \right) \\
&\quad -\frac{1}{2}\left(\frac{x\sigma}{n}\right)^2 n(\phi+2)\left(-\frac{1}{\phi+1} + 1 + \frac{4}{\phi}\right) \\
&\quad +O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= -\frac{1}{2}\frac{(x\sigma)^2}{n}(\phi+2)\left(\frac{3\phi+4}{\phi(\phi+1)} + 1\right) + O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= -\frac{1}{2}\frac{(x\sigma)^2}{n}(\phi+2)\left(\frac{3\phi+4+2\phi+1}{\phi(\phi+1)}\right) + O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= -\frac{1}{2}x^2\sigma^2\left(\frac{5(\phi+2)}{\phi n}\right) + O\left(n\left(x\sigma/n\right)^3\right).
\end{aligned}
$$

(Sketch of the) Proof of Gaussianity

But recall that

$$\sigma^2 = \frac{\phi n}{5(\phi + 2)}.$$

Also, since $\sigma \sim n^{-1/2}$, $n\left(\frac{x\sigma}{n}\right)^3 \sim n^{-1/2}$. So for large n, the $O\left(n\left(\frac{x\sigma}{n}\right)^3\right)$ term vanishes. Thus we are left with

$$\begin{aligned}\log S_n &= -\frac{1}{2}x^2 \\ S_n &= e^{-\frac{1}{2}x^2}.\end{aligned}$$

Hence, as n gets large, the density converges to the normal distribution:

$$\begin{aligned}f_n(k)dk &= N_n S_n dk \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}x^2} \sigma dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.\end{aligned}$$

□

General recurrence: generating functions, method of moments.

Gaps in the Bulk

Distribution of Gaps

For $F_{r_1} + F_{r_2} + \cdots + F_{r_n}$, the gaps are the differences $r_n - r_{n-1}, r_{n-1} - r_{n-2}, \dots, r_2 - r_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

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Can ask similar questions about binary or other expansions: $2024 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3$.

Main Result

Theorem (Distribution of Bulk Gaps (SMALL 2012))

Let $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n+1-L}$ *be a positive linear recurrence of length* L *where* $c_i \geq 1$ *for all* $1 \leq i \leq L$*. Then*

$$P(j) \ = \ \begin{cases} 1-(\frac{a_1}{C_{Lek}})(2\lambda_1^{-1}+a_1^{-1}-3) & : j = 0 \\ \lambda_1^{-1}(\frac{1}{C_{Lek}})(\lambda_1(1-2a_1)+a_1) & : j = 1 \\ (\lambda_1-1)^2\left(\frac{a_1}{C_{Lek}}\right)\lambda_1^{-j} & : j \geq 2. \end{cases}$$

Special Cases

Theorem (Base B Gap Distribution (SMALL 2011))

For base B decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$*, and for* $k \geq 1$, $P(k) = c_B B^{-k}$*, with* $c_B = \frac{(B-1)(3B-2)}{B^2}$.

Theorem (Zeckendorf Gap Distribution (SMALL 2011))

For Zeckendorf decompositions, $P(k) = 1/\phi^k$ *for* $k \geq 2$*, with* $\phi = \frac{1+\sqrt{5}}{2}$ *the golden mean.*

Proof of Bulk Gaps for Fibonacci Sequence

Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1}\frac{n}{\phi^2+1}$.

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Let $X_{i,j} = \#\{m \in [F_n, F_{n+1})$: decomposition of m includes F_i, F_j, but not F_q for $i < q < j\}$.

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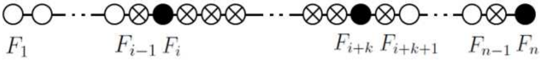
Let $X_{i,j} = \#\{m \in [F_n, F_{n+1})$: decomposition of m includes F_i, F_j, but not F_q for $i < q < j\}$.

$$P(k) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

Calculating $X_{i,i+k}$

How many decompositions contain a gap from F_i to F_{i+k}?



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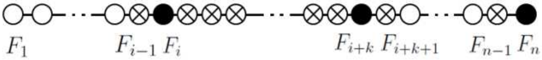

For the indices less than i: F_{i-1} choices. Why? Have F_i as largest summand and follows by Zeckendorf: $\#[F_i, F_{i+1}) = F_{i+1} - F_i = F_{i-1}$.

Calculating $X_{i,i+k}$

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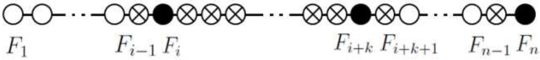

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For the indices greater than $i + k$: $F_{n-k-i-2}$ choices. Why? Shift. Choose summands from $\{F_1, \dots, F_{n-k-i+1}\}$ with $F_1, F_{n-k-i+1}$ chosen. Decompositions with largest summand $F_{n-k-i+1}$ minus decompositions with largest summand F_{n-k-i}.

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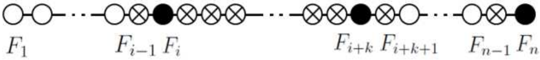

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So total number of choices is $F_{n-k-2-i}F_{i-1}$.

Determining $P(k)$

Recall

$$P(k) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1}\frac{n}{\phi^2+1}} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} F_{n-k-2-i}F_{i-1}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

Binet's formula, sums of geometric series: $P(k) = 1/\phi^k$.

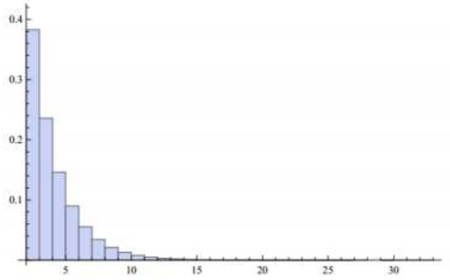

Figure: Distribution of summands in $[F_{1000}, F_{1001})$.

Thoughts on the Computation

Need sum of product of Fibonaccis (Binet's formula, geometric series).

Need average number of gaps/summands (or sum the sum of products).

Can remove contribution from terms within $\log n$ of ends.

Take limit, keeping just main term.

New Approach: Theory of Normalization Constants

If $p(x) = Cg(x)$ is a density and recognize $g(x)$ can get C as integral is 1.

Chi-square distribution: If X is a chi-square distribution with $\nu \geq 0$ degrees of freedom, then X has density

$$f(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu/2-1)} e^{-x/2} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We write $X \sim \chi^2(\nu)$ to denote this.

If $Y_{\nu_1} \sim \chi^2(\nu_1)$ and $Y_{\nu_2} \sim \chi^2(\nu_2)$ are two independent, chi-square random variables, then $Y_{\nu_1} + Y_{\nu_2} \sim \chi^2(\nu_1 + \nu_2)$.

New Approach: Theory of Normalization Constants (cont)

$t = xu$, $t : 0 \to x$ becomes $u : 0 \to 1$.

$$
\begin{aligned}
(f_1 * f_1)(x) &= \int_{-\infty}^{\infty} f_1(t) f_1(x-t) dt \\
&= \int_0^x c_1 t^{-1/2} e^{-t/2} \cdot c_1 (x-t)^{-1/2} e^{-(x-t)/2} dt.
\end{aligned}
$$

The range of integration stops at x as $f_1(x-t)$ is zero if the argument is negative. Simplifying yields

$$
(f_1 * f_1)(x) = c_1^2 e^{-x/2} \int_0^x t^{-1/2} (x-t)^{-1/2} dt.
$$

New Approach: Theory of Normalization Constants (cont)

$$\begin{aligned}(f_1 * f_1)(x) &= c_1^2 e^{-x/2} \int_0^1 (xu)^{-1/2}(x - xu)^{-1/2} x du \\ &= c_1^2 e^{-x/2} \frac{x}{x^{1/2}x^{1/2}} \int_0^1 u^{-1/2}(1-u)^{-1/2} du.\end{aligned}$$

$$(f_1 * f_1)(x) = \begin{cases} \mathcal{C}_1 c_1^2 e^{-x/2} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

New Approach: Gap in the Bulk Distribution

Claim $p_n(g)/p_n(g+1) \to \varphi$, bypassing the difficult calculations.

Essentially lengthen $\mathcal{I}_{\rm middle}$ by one:

$$\frac{p_n(g)}{p_n(g+1)} = \frac{\sum_{i=1}^{n-g} F_{i-1} F_{n-i-g-2}}{\sum_{i'=1}^{n-(g+1)} F_{i'-1} F_{n-i'-(g+1)-2}}.$$

Numerator one more summand, summands differ by 1 in index.

New Approach: Gap in the Bulk Distribution (cont)

Remove $\log n$ indices near ends.

Indices of $F_{n-i-g-2}$ and $F_{n-i-(g+1)-2} = F_{n-i-g-3}$ are large, from Binet's Formula

$$F_{n-i-g-2} \approx \varphi F_{n-i-g-3}.$$

$p_n(g)/p_n(g+1) \to \varphi$; if limiting distribution is discrete geometric with parameter φ.

Future Work

SMALL 2024

Gaps of General Recurrences.

Longest Gap.

Gaussianity for Number of Summands.

Benfordness of Summands.

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Thank you!

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