Variants on the Zeckendorf Game

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Introduction and Definitions

Zeckendorf Decomposition:

- Zeckendorf's Theorem: Every integer n can be written as a unique sum of non-adjacent Fibonacci Numbers.
- For example:

$$\begin{aligned} 10 &= 2+8 \\ 27 &= 1+5+21 \\ 66 &= 55+8+3 \end{aligned}$$

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- Players take turns making one of two moves.
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    - Splitting two coins from F_n into one coin each on F_{n-2} and F_{n+1}

$$2Fn \rightarrow F_{n-2} \wedge F_{n+1}.$$

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- The game terminates when the full Zeckendorf decomposition is reached.

Zeckendorf Game Example: $n = 6$

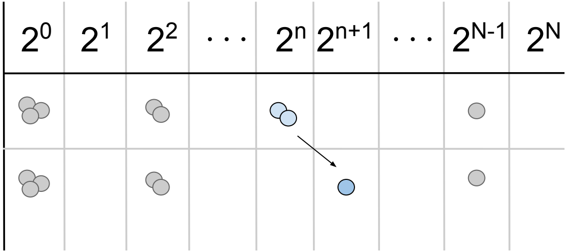
n=6	1	2	3	5	8
	6				
P1	4	1			
P2	3	0	1		
P1	1	1	1		
P2	1	0	0	1	

- Baird-Smith, Epstein, Flint, Miller proved nonconstructively that for $n > 2$, Player 2 has the winning strategy.

The Binary Game

- Game starts with n coins in pile marked with 1, and empty piles marked $2, 4, 8, \dots 2^N$.
- Every turn: Player can remove 2 coins from a pile marked 2^n, and add a coin to the pile marked 2^{n+1}:

$$\{2^n \wedge 2^n \;\to\; 2^{n+1}\}.$$



The Binary Game

- Natural analog: First player to obtain the full binary decomposition wins.
- Every turn removes one coin and at the end we have $B(n)$ coins. Therefore the game has a determined length $n - B(n)$.
- Winner is trivially obtained by the parity of that length:
  - If $(n - B(n)) \equiv 0 \bmod 2$ then Player 2 wins.
  - If $(n - B(n)) \equiv 1 \bmod 2$ then Player 1 wins.

Binary Game Example: $n = 6$

Below, we let $n = 6$, and play the binary game to the binary decomposition.
The number of moves is $n - B(n) = 6 - 2 = 4$, and thus Player 2 wins.

n=6	2^0	2^1	2^2
	6		
P1	4	1	
P2	2	2	
P1	2	0	1
P2	0	1	1

Highest Summand Binary Game

Highest Summand Variant: The player to make the first move to 2^N where

$$2^N \leq n < 2^{N+1}.$$

is the winner.

- Exploring many possible variants, we found that this is the only binary game where the winner is not predetermined.

Winning Strategy

We define $f(n) : \mathbb{N} \to \{0, 1\}$ to be the function determining which player wins a game with n initial coins, where 1 corresponds with the first player to move winning, and 0 with the second.

Theorem 1

Let

$$n := \sum_{i=0}^{N} b_i 2^i$$

where each $b_i \in \{0, 1\}$. We have

$$f(n) = n - B(n) - b_{N-1} \mod 2.$$

Proof of Theorem 1

Note that in any position $(\ldots, a, 1, 0)$ with $a \geq 2$, if any player moves a second coin to the $(N-1)^{\text{th}}$ spot, the next player wins on the following move.

Define X to be the state where the game is in the form $(b_0, \ldots, b_{N-3}, a, 1, 0)$ for some $a \geq 2$.

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The following hold:

1. We must eventually reach state X.
2. It takes $n - B(n) - 2 - b_{N-1}$ moves to reach state X.

Proof of Theorem 1

Claim 1:

- Because the game must terminate, we must reach some state $(\ldots, 2, 1, 0)$.

- The first $N-2$ positions always make up some state in a smaller binary game.

- Any binary game must terminate in the full binary decomposition, so either:
  - each of the first $N-2$ positions only have 1 or 0, or
  - there is another move to be made from those positions.

Proof of Theorem 1

Claim 2:

- We know it takes $n - B(n)$ moves to reach the full decomposition D.

State D	State X	Number of Moves from X to D
$(\ldots, 0, 0, 1)$	$(\ldots, 2, 1, 0)$	2
$(\ldots, 1, 0, 1)$	$(\ldots, 3, 1, 0)$	2
$(\ldots, 0, 1, 1)$	$(\ldots, 4, 1, 0)$	3
$(\ldots, 1, 1, 1)$	$(\ldots, 5, 1, 0)$	3

- If $b_{N-1} = 0$, it takes 2 additional moves to reach D, so it takes $n - B(n) - 2$ moves to reach X.
- If $b_{N-1} = 1$, it will take $n - B(n) - 3$ moves to reach X.

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- If $b_{N-1} = 0$, it takes 2 additional moves to reach D, so it takes $n - B(n) - 2$ moves to reach X.
- If $b_{N-1} = 1$, it will take $n - B(n) - 3$ moves to reach X.

So, since the game always terminates 2 moves after we reach X, the game takes $n - B(n) - b_{N-1}$ moves.

Recursion

- For any odd number $2n+1$, the first bit is 1 and the rest are identical to $2n$. Therefore:
  - $2n+1$ and $2n$ have opposite parity,
  - $B(2n+1)$ and $B(2n)$ have opposite parity.

$$f(2n+1) \;=\; f(2n).$$

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- $2n$ has the same decomposition as n, but every bit is shifted by one place. Since the parity matches for the rest of the formula, the winner is determined by the parity of n:

$$f(2n) \;=\; \begin{cases} f(n) & n \equiv 0 \mod 2 \\ f(n)+1 \mod 2 & n \equiv 1 \mod 2 \end{cases}.$$

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Who wins?

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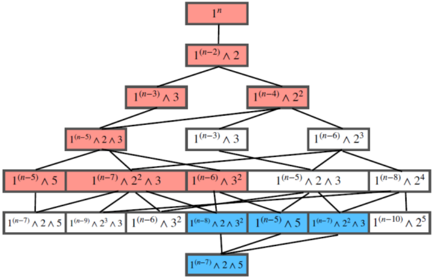
Original Game: Player 2 always wins

- Non-Constructive Proof
- Relies on Player 2 “stealing" Player 1’s strategy.



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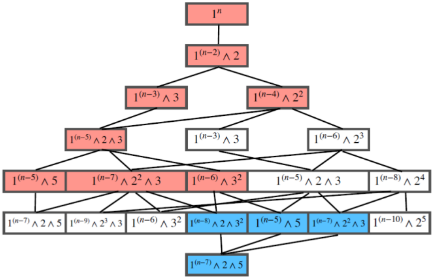

Highest Summand Zeckendorf Game

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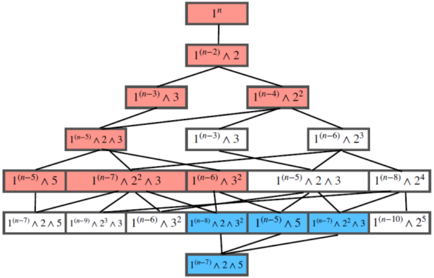
Proof applies to many variants where winning is conditional on certain game state.



Non-Constructive Proof: Second Player Wins

Theorem 2

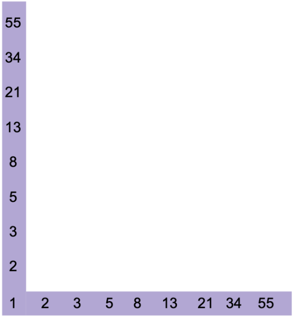
Suppose we play the Zeckendorf game where a player wins if S occurs. If S does not occur on the tree in the proof of Theorem 1.7 in [Bai+18], and S is only conditional on the game state, then Player 2 wins for $n \geq 9$.

Proof.

The proof of the winner of the highest summand Zeckendorf game follows the same structure as the proof of the game resulting in the Zeckendorf decomposition. □

2-Branch Game

- We expand the two-player game to a 2-dimensional case, played on an L-shaped board.



2-Dimensional 2-Player Board

2-Branch Game

- The legal moves from the original game are the same:

$$F_{n-1} \wedge F_n \;\rightarrow\; F_{n+1}$$

  and

$$2F_n \;\rightarrow\; F_{n-2} \wedge F_{n+1}.$$

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Conjecture 1

In the 2-dimensional version of the Zeckendorf game, the second player always has a winning strategy.

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Thank You!

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