Zeckendorf-Niven Numbers in Arithmetic Progressions

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Overview

Previous Work - Niven Numbers

Definition

We call a natural number $n \in N$ a *Niven number* if it is divisible by the sum of its digits.

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Example:

135 is *Niven* as $9 \mid 135$

89 is not *Niven* as $17 \not| 89$

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Example:

$$34 = 34, \quad 12 = 8 + 3 + 1, \quad 63 = 55 + 8$$

Zeckendorf-Niven number

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Example:

$$159 = 144 + 13 + 2 \text{ is } \textit{Zeckendorf-Niven}$$

$$7 = 5 + 2 \text{ is not } \textit{Zeckendorf-Niven}$$

Previous Work

Theorem (Grundman 2007)

- Any sequence of 5 or more consecutive Zeckendorf-Niven numbers are a subsequence of 1,2,3,4,5,6.
- There are infinitely many sequences of 4 consecutive Zeckendorf-Niven numbers.

Arithmetic Progressions

Definition

An *arithmetic progression*, $(a_n)_{n=0}^{\infty}$ is a sequence such that there exists $d, b \in \mathbb{N}$ where

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(This is an example of a *3-AP*)

Zeckendorf-Niven Numbers in 2-APs

Theorem (Grundman 2007)

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Theorem

- Any sequence of 8 or more Zeckendorf-Niven numbers in 2-AP are a subsequence of 2, 4, 6, 8, 10, 12, 14, 16, 18.
- There are infinitely many 2-AP of 5 Zeckendorf-Niven numbers.

Results in F_a-APs

Theorem

For any Fibonacci F_a, there does not exist a sequence $n, n + F_a, n + 2F_a, n + 3F_a$ such that $S_z(n) = S_z(n + F_a) = S_z(n + 2F_a) = S_z(n + 3F_a)$.

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Example:

$$14 = 13+1, \quad 16 = 13+3, \quad 18 = 13+5, \quad 20 = 13+5+2$$

Results in F_a-APs

Lemma

If $S_z(n) = S_z(n+k) = y$ where $k \geq 1$ and y is not a factor of k, then n and $n+k$ cannot both be Zeckendorf-Niven.

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Example:

$$48 = 34 + 13 + 1, \quad 50 = 34 + 13 + 3$$

Results in F_a-APs

Theorem

For any Fibonacci $F_a \geq 2$, there exists an infinite number of Zeckendorf-Niven sequences $n, n + F_a, n + 2F_a$ such that $S_z(n) = S_z(n + F_a) = S_z(n + 2F_a) = y$ where $y \geq 2$ is a factor of F_a.

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Example for $F_a = 3$ and $y = 3$:

$$324 = 233 + 89 + 2, \;\; 327 = 233 + 89 + 5, \;\; 330 = 233 + 89 + 8$$

Zeckendorf-Niven numbers in every AP

Pisano Period Lemma

For every $n \in \mathbb{N}$ the sequence $(F_k)_{k=0}^{\infty}$ is periodic modulo n.

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Example: For $n = 4$:

$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 1, F_6 = 0, F_7 = 1,$
so $\pi(4) = 6$.

Zeckendorf-Niven numbers in every AP

Lemma

For every $n, m \in \mathbb{N}$, $n \mid F_{m\pi(n)}$.

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Lemma

For every $b, m, d \in \mathbb{N}$, there exists $n \in \mathbb{N}$, such that $S_z(b) + m = S_z(nd + b)$.

Core of the proof: $S_z(b) + 1 = S_z(F_{N\pi(d)} + b)$

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The main idea of the result:

$$N = \sum_{i=1}^{m} F_{a_i} \to F_{a_m+\pi(d)} + \sum_{i=1}^{m-1} F_{a_i} \to F_{a_m+k\pi(d)} + \sum_{i=1}^{m-1} F_{a_i} = N',$$

$\gcd(\pi(m), \pi(d)) = 1$.

Zeckendorf-Niven numbers in every AP

Lemma (Koshy 2001)

For all $n \in \mathbb{N}$ we have $\pi(F_{2n}) = 4n$ and $\pi(F_{2n+1}) = 8n + 4$.

Zeckendorf-Niven numbers in every AP

Lemma (Koshy 2001)

For all $n \in \mathbb{N}$ we have $\pi(F_{2n}) = 4n$ and $\pi(F_{2n+1}) = 8n + 4$.

Theorem

For every $d, b \in \mathbb{N}$ there is infinitely many Zeckendorf-Niven numbers in the sequence $\{nd + b\}_{n=0}^{\infty}$.

Generalization To The Lucas Sequence

Lucas sequence: $L_{n+1} = L_n + L_{n-1}$;

$$L_0 = 2, L_1 = 1, L_2 = 3, L_3 = 4, \dots .$$

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Theorem

For every $d, b \in \mathbb{N}$ there is infinitely Lucas-Niven number in the sequence $\{nd + b\}_{n=0}^{\infty}$.

Future Work

- Generalize the previous theorem for any decomposition based on an positive linear recurrence sequence (PLRS).

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}$$

- Generalize theorems on consecutive numbers with a non-Fibonacci difference and with the same number of coefficients in their Zeckendorf decompositions.
- Generalize the bounds for the number of consecutive Zeckendorf-Niven numbers in an arbitrary arithmetic progression.

References

Grundman, H. G.
Consecutive Zeckendorf-Niven and lazy-Fibonacci-Niven numbers.
The Fibonacci Quarterly, 45(3), 272-276.

Koshy, T.
Fibonacci and Lucas Numbers with Applications (2001), 196-210. and 415-423.

Wall, D.D.
Fibonacci series modulo m.
Math. Monthly **67** (1960), 525-532.