Black Hole Zeckendorf Games

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Zeckendorf Introduction

Zeckendorf Decompositions

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Standard Zeckendorf Game

Define a game using the Fibonacci numbers as columns that moves from representing a number n as a sum of 1's to a Zeckendorf Decomposition using 3 different moves.

Zeckendorf Game Setup

Setup: The game is played on a board with columns corresponding to each of the Fibonacci numbers, indexing so that the 1st column corresponds with $F_1 = 1$, the 2nd column corresponds with $F_2 = 2$ and the m^{th} column corresponds with F_m, the m^{th} Fibonacci number. All n pieces begin in the F_1 column.

Zeckendorf Game Moves, Part 1

Gameplay: Players alternate, selecting their moves from the following.

1. Adding consecutive terms: If the board contains pieces in both F_i and F_{i-1} columns, players can remove one piece from each column to add as one piece in the F_{i+1} column.

F_1	F_2	F_3	F_4
n	1	1	0

$\Longrightarrow$

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2. Merging 1's: If the board contains more than one piece in the F_1 column, players can remove two pieces from the F_1 column to merge as one piece in the F_2 column.

F_1	F_2	F_3	F_4
n	0	0	0

$\Longrightarrow$

F_1	F_2	F_3	F_4
$n-2$	1	0	1

Zeckendorf Game Moves, Part 2

- Splitting: If the board contains more than one piece in the F_2 column, players can split two pieces from the F_2 column to place one piece in each of F_1 and F_3. For $i \geq 3$, players can split two pieces in the F_i column to place one in each of F_{i-2} and F_{i+1}.

F_1	F_2	F_3	F_4
0	0	2	0

$\Longrightarrow$

F_1	F_2	F_3	F_4
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$\Longrightarrow$

F_1	F_2	F_3	F_4
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Slightly different for F_2.

F_1	F_2	F_3	F_4
0	2	0	0

$\Longrightarrow$

F_1	F_2	F_3	F_4
1	0	1	0

Winning: The last player to move wins.

Black Hole Variation

Black Hole Definition

Works as a normal Zeckendorf Game, but we remove all pieces on and after a designated Fibonacci number

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Adds an additional phase before a Black Hole Zeckendorf game begins, where players take turns choosing whether to place a piece in the first or last available column.

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Prior Work and Our Goals

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- Additionally, they showed a non-constructive proof that player 2 has a winning strategy for $n \neq 2$.
- Constructive strategy for the regular Zeckendorf game still unknown.
- We hoped to find a constructive solution, ended up blundering into other interesting mathematics.

Notation

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- Solutions are based on modular arithmetic, we also describe game states in terms of α, β, γ and k_1, k_2, k_3, where α and k_1 correspond with the F_1 column, β and k_2 correspond with the F_2 column, and γ and k_3 with the F_3 column.

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- For example, we might describe a board state with a black hole on F_3 as (a, b) or as $(3\alpha + k_1, 3\beta + k_2)$.

Game Trees

Throughout we sketch out the various moves available to players using trees, as below:

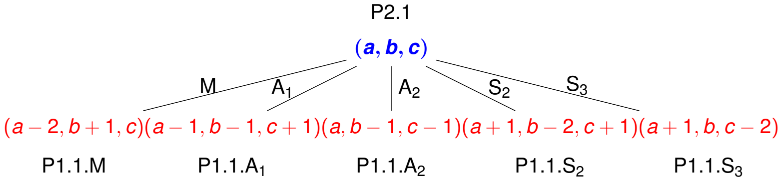

Figure: Example Game Tree for a Setup (a, b, c),

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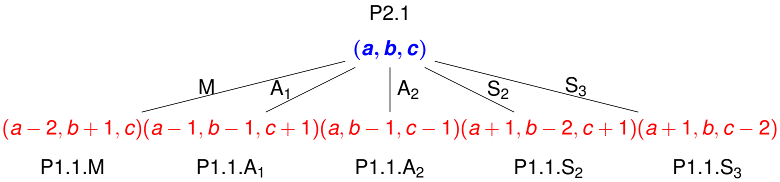

Figure: Example Game Tree for a Setup (a, b, c),

Red denotes player 1, while ***Blue*** denotes player 2. Additionally, M, A, and S denote the moves, with subscripts for the columns. These moves are listed left to right, though not every tree has all 5 possibilities.

Black Hole on F_3

Theorem: Cashman, Miller, Son, and Shuffelton

The winner of all possible games for a Black Hole on F_3 can be determined based on the value of a and b modulo 3. We find that Player 2 wins (a, b) for all $a \equiv b \equiv 0$, $a \equiv 0, b \equiv 1$ or $a \equiv 1, b \equiv 0$. Player 1 wins for any other setup.

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- Next, we consider a few more general board states without an empty or nearly empty column. Induction and eventually reducing to an empty column case gives us a result for several cases.

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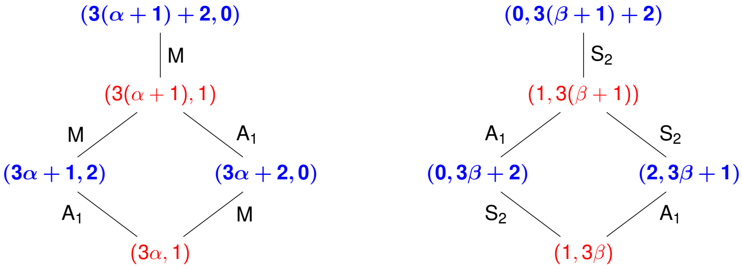
- Next, we consider a few more general board states without an empty or nearly empty column. Induction and eventually reducing to an empty column case gives us a result for several cases.
- To finish every case, we make a few moves that force them into one of the already solved cases.

Game Trees for F_3



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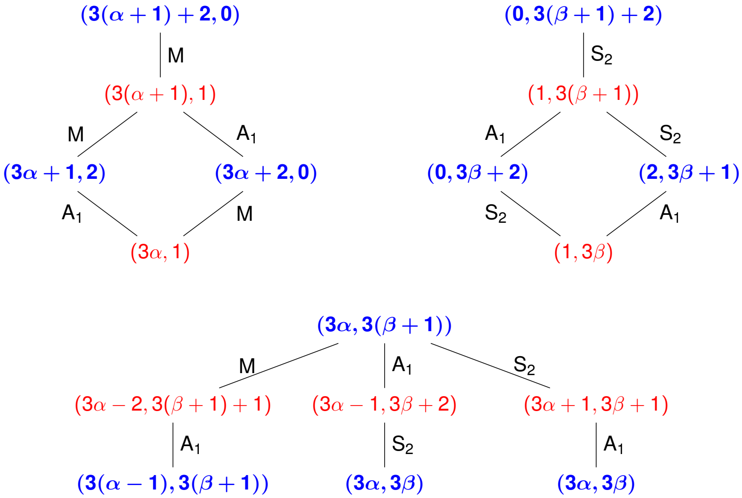

Empty Board on F_3

Cashman et al.

Starting from an empty board, Player 1 has a constructive strategy for winning for any $n \equiv 1, 2, 3, 6, 8 \pmod 9$. Player 2 has a constructive strategy for winning for any $n \equiv 0, 4, 5, 7 \pmod 9$.

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Our proof uses a move mirroring strategy - after the first few moves, it is optimal for the eventually winning player to mirror their opponent, placing in the opposite column.

Black Hole on F_4 Results

	$a \equiv 0 \pmod 3$	$a \equiv 1 \pmod 3$	$a \equiv 2 \pmod 3$
$c \equiv 0 \pmod 4$	$\boldsymbol{\alpha \geq \gamma}$ $\alpha \leq \gamma - 1$	$\boldsymbol{\forall \alpha, \gamma}$	$\boldsymbol{\alpha \geq \gamma + 1}$ $\alpha \leq \gamma$
$c \equiv 1 \pmod 4$	$\boldsymbol{\alpha \geq \gamma - 1}$ $\alpha \leq \gamma - 2$	$\boldsymbol{\forall \alpha, \gamma}$	$\boldsymbol{\alpha \geq \gamma}$ $\alpha \leq \gamma - 1$
$c \equiv 2 \pmod 4$	$\forall \alpha, \gamma$	$\alpha \geq \gamma + 1$ $\boldsymbol{\alpha \leq \gamma}$	$\forall \alpha, \gamma$
$c \equiv 3 \pmod 4$	$\forall \alpha, \gamma$	$\alpha \geq \gamma$ $\boldsymbol{\alpha \leq \gamma - 1}$	$\forall \alpha, \gamma$

Winners for board setups $(a, 0, c)$ in an F_4 Black Hole Zeckendorf Game. Player 2 wins are depicted in bold blue, and Player 1 wins are depicted in red.

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- Forcing moves still works if the board starts with all but the F_1 or F_3 column empty.
- First, we show that certain states win non-constructively, so as to guide constructive strategies to them.
- Finally, we go back and use our earlier findings to provide constructive strategies.

Further Details and Observations

- We frequently care about whether $\alpha > \gamma$ due to the game coming down to whether the F_1 or F_3 column empties first.

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- We frequently care about whether $\alpha > \gamma$ due to the game coming down to whether the F_1 or F_3 column empties first.
- The non-constructive portions mostly rely on parity switching, just as in the proof of Baird-Smith et al. that player 2 wins most regular Zeckendorf games.
- Both constructive and non-constructive parts re-use substantial tools from the black hole on F_3 game, particularly induction on α and γ. We also occasionally use double induction on both.

Empty Board on F_4

Theorem, Cashman et al.

Player 2 constructively wins an Empty Board F_4 Black Hole Zeckendorf game for $n \equiv 0, 2, 4, 6, 9, 11, 13 \pmod{16}$ when $n \neq 2, 32$, and also wins $n = 17, 47$.
Player 1 instead wins for $n \equiv 1, 3, 5, 7, 8, 10, 12, 14, 15 \pmod{16}$ when $n \neq 17, 47$, and also wins $n = 2, 32$.

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Main element of proof is mirror moving allowing for players to force advantageous setups.
The exceptions for $n = 2, 17, 32, 47$ stem from them being cases where $\alpha \leq \gamma$.

Future Work

1. A full consideration of the empty board game with black hole on F_4 - we were unable to analyze a starting board of (a, b, c) for non-zero b.

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2. Expanding to black holes on higher columns would also be valuable. Unfortunately, we loose a lot of tools such as forcing, even for a black hole on F_5.
3. Figuring out either ways to connect this work back to general Zeckendorf games, or some other system or familiar game.

Acknowledgements

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Thank you!

Any questions?

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