Second Moment Calculations for One-Parameter Families of Elliptic Curves

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1 Introduction

2 Computation of a_p

3 Computational Investigation

4 Normalized Second Moment

5 Algebraic Geometry

Elliptic Curves

An elliptic curve E is a genus 1 smooth projective curve with a distinguished point $\mathcal{O}_E$ (a "point at infinity").

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$$E(K) \coloneqq \{(x, y) \in K^2 : y^2 = x^3 + Ax + B\} \cup \{\mathcal{O}_E\},$$

with discriminant $\Delta(E) \coloneqq -16(4A^3 + 27B^2) \neq 0$.

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Theorem (Mordell-Weil 1922)

The set $E(\mathbb{Q})$ is a finitely generated abelian group whose identity element is $\mathcal{O}_E$:

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E(\mathbb{Q})_{\mathrm{tors}},$$

and we call r the rank of $E(\mathbb{Q})$.

Elliptic Curves mod p

From $E/\mathbb{Q}$ we obtain an infinite family of elliptic curves mod p, one for each $p \nmid \Delta(E)$:

- Reduction mod p is a group homomorphism $E(\mathbb{Q}) \longrightarrow E(\mathbb{F}_p)$;
- The expected number of solutions to $E_p := E(\mathbb{F}_p)$ is $p + 1$.

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Theorem (Hasse 1933)

For $a_p := p + 1 - \#E(\mathbb{F}_p)$ we have that $|a_p| \leq 2\sqrt{p}$.

Hasse-Weil L-function

If $p \mid \Delta(E)$, let $\alpha_p \coloneqq 0, 1$, or -1 based on the type of singularity of E mod p.

Definition

The Hasse-Weil L-function of E is defined by

$$L(s, E) \coloneqq \prod_{p \mid \Delta(E)} \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \prod_{p \nmid \Delta(E)} \left(\frac{p^s + p^{1-s} - a_p}{p^s}\right)^{-1},$$

which converges uniformly and absolutely on compact subsets of $\{s \in \mathbb{C} : \mathrm{Re}(s) > 3/2\}$ and has an analytic continuation to all of $\mathbb{C}$.

The L-function $L(s, E)$ packages the information from the a_p's.

Birch and Swinerton-Dyer

Birch and Swinerton-Dyer 1950s: Computational experiments calculating

$$\pi_E(X) := \prod_{p \leq X, p \nmid \Delta(E)} \frac{\#E(\mathbb{F}_p)}{p} = \prod_{p \leq X, p \nmid \Delta(E)} \frac{p+1-a_p}{p}$$

led to the following conjecture.

Conjecture (Birch and Swinnerton-Dyer)

If $E/\mathbb{Q}$ has rank r, then there is some constant C such that $\pi_E(X) \sim C \log(X)^r$.

Birch and Swinerton-Dyer

Although the Euler product may not converge for $s = 1$, formally evaluating $L(s, E)$ at $s = 1$

$$L(1, E) \text{ “} = \text{”} \left(\prod_{p \mid \Delta(E)} \left(1 - \frac{\alpha_p}{p}\right) \prod_{p \nmid \Delta(E)} \frac{p + 1 - a_p}{p} \right)^{-1}.$$

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Idea: Larger $\#E(\mathbb{F}_p) = p + 1 - a_p$ on average, i.e., negative bias in the a_p's, should imply $L(s, E)$ tends to zero faster as $s \to 1$.

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Conjecture (Birch and Swinnerton-Dyer)

If $E/\mathbb{Q}$ has rank r, then

$$r \;=\; \mathrm{ord}_{s=1} L(s, E).$$

Connecting the two conjectures

Theorem (Goldfeld)

Suppose $\pi_E(X) \sim C\log(X)^r$. *Then* $r = \mathrm{ord}_{s=1}L(s,E)$ *and*

$$\lim_{s\to 1}\frac{L(s,E)}{(s-1)^r} = \frac{\sqrt{2}\mathrm{e}^{r\gamma}}{C}\prod_{p|\Delta(E)}(1-\alpha_p p^{-1})^{-1}.$$

In particular, if $r = 0$ *then*

$$L(1,E) = \sqrt{2}\left(\prod_{p|\Delta(E)}\left(1-\frac{\alpha_p}{p}\right)\prod_{p\nmid\Delta(E)}\frac{p+1-a_p}{p}\right)^{-1}.$$

One-parameter families

Let $\mathscr{E} \to \mathbb{P}^1$ be a non-split elliptic surface over $\mathbb{Q}$ with

$$\mathscr{E} : y^2 = x^3 + A(T)x + B(T),$$

where $A(T), B(T) \in \mathbb{Z}[T]$ and $4A(T)^3 + 27B(T)^2 \neq 0$. Almost all specializations $T = t \in \mathbb{Z}$ results in an elliptic curve E_t. This is a *one-parameter family* $\{E_t\}$ of elliptic curves over $\mathbb{Q}$.

- The *rank of a family* is the minimum rank r that appears infinitely often in E_t.

n-th Moments

Definition (n-th Moments of a 1-parameter Family of Elliptic Curves)

Let $a_t(p) \coloneqq p + 1 - \#E_t(\mathbb{F}_p)$. Then the n-th moment is given by

$$\mathscr{A}_{n,\mathscr{E}}(p) \;\coloneqq\; \sum_{t=0}^{p-1} a_t(p)^n.$$

Note, we do not normalize by $1/p$.

Moments and the arithmetic properties of elliptic curves

Theorem (Nagao, Rosen-Silverman 1998)

Assuming Tate's conjecture holds for $\mathscr{E}$, we have that

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\leq X} \mathscr{A}_{1,\mathscr{E}}(p) \frac{\log p}{p} \;=\; -\mathrm{rank}\mathscr{E}(\mathbb{Q}(T)).$$

Assuming Tate's conjecture (known for $\mathscr{E}$ if degrees of A, B sufficiently small):

- the rank is determined by the first moment;
- $\sum_{p\leq X} \log p \sim X$ so if $\mathscr{A}_{1,\mathscr{E}}(p) = -rp + O(1)$, then $\mathrm{rank}\mathscr{E}(\mathbb{Q}(T)) = r$;
- there is a negative bias in $a_t(p)$.

The Second Moment

The j-invariant of an elliptic surface $\mathscr{E}$ is

$$j(T) \;:=\; 1728\frac{4A(T)^3}{4A(T)^3+27B(T)^2}.$$

Second Moment Asymptotic (Michel, 1995)

For an elliptic surface $\mathscr{E}$ with $j(T)$ non-constant, we have

$$\mathscr{A}_{2,\mathscr{E}}(p) \;=\; p^2+O(p^{3/2}),$$

with lower order terms of size $p^{3/2}, p, p^{1/2}$, and 1.

Bias Conjecture

Question

Do lower order terms in the asymptotic second moment expansion also have a negative bias?

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Conjecture (S.J. Miller '02)

The largest lower order term in the second moment expansion which does not average to 0 is on average negative.

- *Excess rank problem*: disagreement between expected and observed average rank.
- Miller: negative bias in lower order terms increases bound for average rank obtained via 1-level density of zeros near central point.

Strong Bias Conjecture

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The largest lower term in the second moment expansion which does not average to 0 is negative for all but finitely many p.

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The largest lower term in the second moment expansion which does not average to 0 is negative for all but finitely many p.

- SMALL '23 disproved the strong bias conjecture with $x^3 + x + T^3$.
- For $p \equiv 2 \bmod 3$, there exists an automorphism $x^3 + x + T^3 \longmapsto x^3 + x + T$.
- The family $x^3 + x + T$ has second moment $p^2 + p$ for $p \equiv 2 \bmod 3$.

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Legendre Symbols

The Legendre symbol is

$$\left(\frac{a}{p}\right) := \begin{cases} 0 \text{ when } a = 0, \\ 1 \text{ when } a \neq 0 \text{ and square}, \\ -1 \text{ otherwise.} \end{cases}$$

Observation

When p is odd and for a given x, the number of solutions to $y^2 = x^3 + ax + b$ mod p is given by

$$1 + \left(\frac{x^3 + ax + b}{p}\right).$$

Legendre Symbols

Lemma

For p odd, then the elliptic curve given by $y^2 = x^3 + ax + b$ satisfies

$$a_p = -\sum_x \left(\frac{x^3 + ax + b}{p} \right).$$

Legendre Symbols

Lemma

For p odd, then the elliptic curve given by $y^2 = x^3 + ax + b$ satisfies

$$a_p = -\sum_x \left(\frac{x^3 + ax + b}{p} \right).$$

Consequently, we can calculate the second moment via

$$\mathscr{A}_{2,\mathscr{E}} = \sum_t \sum_{x,y} \left(\frac{x^3 + A(t)x + B(t)}{p} \right) \left(\frac{y^3 + A(t)y + B(t)}{p} \right).$$

In general, explicit formulas are difficult, however, it is possile in some special cases.

Known Expansions

Essentially, the only know non-trivial second moment expansions are as follows:

Family	$A_{1,\mathscr{E}}(p)$	$A_{2,\mathscr{E}}(p)$
$y^2 = x^3 + 2^4(-3)^3(9T+1)^2$	0	$\begin{cases} 2p^2-2p & p \equiv 2 \bmod 3 \\ 0 & p \equiv 1 \bmod 3 \end{cases}$
$y^2 = x^3 \pm 4(4T+2)x$	0	$\begin{cases} 2p^2-2p & p \equiv 1 \bmod 4 \\ 0 & p \equiv 3 \bmod 4 \end{cases}$
$y^2 = x^3 + (T+1)x^2 + Tx$	0	$p^2 - 2p - 1$
$y^2 = x^3 + x^2 + 2T + 1$	0	$p^2 - 2p - 3$
$y^2 = x^3 + Tx^2 + 1$	$-p$	$p^2 - n_{3,2,p}p - 1 + c_{3/2}(p)$
$y^2 = x^3 - T^2x + T^2$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 - T^2x + T^4$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 + Tx^2 - (T+3)x + 1$	$-2c_{p,1;4}p$	$p^2 - 4c_{p,1;6}p - 1$

where $c_{p,a;m} = 1$ if $p \equiv a \bmod m$ and 0 otherwise; $n_{3,2,p}$ is the number of cubes root of 2 mod p; $c_\alpha(p)$ are certain Legendre sums multiplied by p.

Tractable Cases

Closed form expressions via Legendre sums are only tractable in special cases: explicit calculations made by reducing to lower degree polynomials $A(T), B(T)$, and using the identities for linear and quadratic Legendre sums

$$\sum_x \left(\frac{ax+b}{p}\right) = \mathbf{1}_{a=0} p \left(\frac{b}{p}\right)$$

and

$$\sum_x \left(\frac{ax^2+bx+c}{p}\right) = \left(\mathbf{1}_{b^2=4ac} ap - 1\right)\left(\frac{a}{p}\right).$$

Computational Investigation of the Bias Conjecture

- S.J. Miller formulated Bias Conjecture after drawing inspiration from proof of Nagao conjecture, computations for low-degree non-generic families.
- To test Bias Conjecture, need to study families with higher degree polynomials $A(T), B(T)$ to observe generic families.
- Computationally investigate families:
  - generate database of a_p values (primes up to $250,000$);
  - study second moments of families with $A(T), B(T)$ of degree ≤ 5.

Automorphisms

Consider the elliptic curve $\mathscr{E}\colon\ y^2 = x^3 + ax + b$. Letting $y = \ell^3 y'$ and $x = \ell^2 x'$ yields

$$\ell^6 y'^2 \ = \ \ell^6 x'^3 + \ell^2 a x' + b.$$

Dividing by ℓ^6, we conclude

$$\mathscr{E}\colon\ y'^2 \ = \ x'^3 + \frac{a}{\ell^4} x' + \frac{b}{\ell^6}.$$

Hence, these two curves are automorphic to each other and have the same number of $\mathbb{F}_p$ points.

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Hence, these two curves are automorphic to each other and have the same number of $\mathbb{F}_p$ points. Thus, it suffices to compute the values of a_p for one a in each quartic residue class. For p odd,

$$\#\text{ of quartic residue classes} = \begin{cases} 5 & p \equiv 1 \pmod 4 \\ 3 & p \equiv 3 \pmod 4. \end{cases}$$

Automorphisms $(a = 0)$

Example

The curves $y^2 = x^3 + 8x + 3$ *and* $y^2 = x^3 + 2x + 9$ *are automorphic mod* 11 *with* $\ell = 3$.

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The curves $y^2 = x^3 + 3$ and $y^2 = x^3 + 9$ are automorphic mod 11 with $\ell = 3$.

This suggests that when $a = 0$, we only need to store 6 different a_p values, one for each sextic residue class of b. Hence,

$$\#\text{ of } a_p \text{ we need to store } = \begin{cases} 4p+6 & p \equiv 1 \pmod 4 \\ 2p+6 & p \equiv 3 \pmod 4. \end{cases}$$

Automorphisms $(a = 0)$ cont.

When $p \not\equiv 1 \pmod 3$, $x \mapsto x^3$ is an bijection. Hence

$$\sum_x \left(\frac{x^3 + b}{p}\right) = \sum_x \left(\frac{x + b}{p}\right) = 0.$$

Therefore, we can compute and store fewer a_p:

$$\#\text{ of } a_p \text{ we need to store } = \begin{cases} 4p & p \equiv 1 \pmod 4 \\ 2p & p \equiv 3 \pmod 4 \end{cases} + \begin{cases} 6 & p \equiv 1 \pmod 3 \\ 0 & p \equiv 0, 2 \pmod 3). \end{cases}$$

Automorphisms $(\sqrt{-1})$

Suppose there is some $i \in \mathbb{F}_p$ such that $i^2 = -1$. Then, put $y = iy'$ and $x = -x$, and thus

$$\begin{aligned} y^2 &= x^3 + ax + b \\ (iy')^2 &= -x^3 - ax + b \\ y'^2 &= x^3 + ax - b. \end{aligned}$$

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$$\begin{aligned} y^2 &= x^3 + ax + b \\ (iy')^2 &= -x^3 - ax + b \\ y'^2 &= x^3 + ax - b. \end{aligned}$$

Hence, we need to compute a_p for $0 \leq b < p/2$. Thus

$$\#\text{ of } a_p \text{ we need to store } = \begin{cases} 2p+2 & p \equiv 1 \pmod 4 \\ 2p & p \equiv 3 \pmod 4 \end{cases} + \begin{cases} 6 & p \equiv 1 \pmod 3 \\ 0 & p \equiv 0, 2 \pmod 3). \end{cases}$$

N.b. this slide has not been implemented due to our code already being sufficiently fast.

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Normalized Second Moment

- By Michel's theorem, there are $\alpha(p), \beta(p)$ both $O(1)$ such that

$$\mathscr{A}_{2,\mathscr{E}}(p) \;=\; p^2 + \alpha(p)p^{3/2} + \beta(p)p + O(p^{1/2}).$$

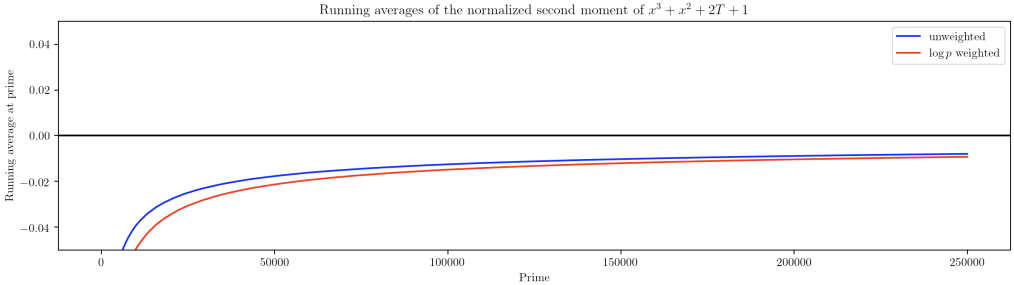
- We compute the *normalized second moment* of $\mathscr{A}_{2,\mathscr{E}}(p)$, which we define to be

$$\mathscr{B}_{2,\mathscr{E}}(p) \;:=\; \frac{\mathscr{A}_{2,\mathscr{E}}(p) - p^2}{p^{3/2}}$$

and graph its running average and $\log p$-weighted running average to find potential positive bias families to investigate further with algebraic geometry.

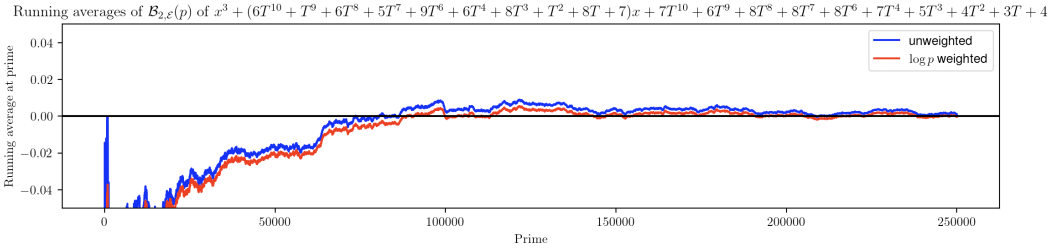
A Family with Known Negative Bias

For $\mathscr{E} : y^2 = x^3 + x^2 + 2T + 1$, we know that $\mathscr{A}_{2,\mathscr{E}}(p) = p^2 - 2p - 3$ has negative bias.

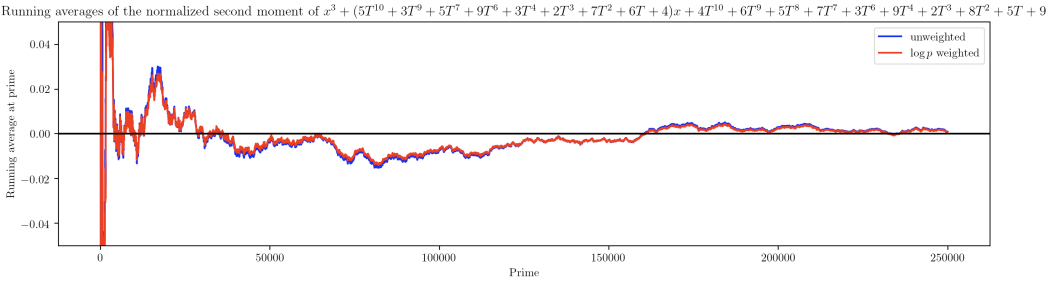


SMALL '24 Generic Random Family 1

We sampled coefficients iid from $\{0, 1, \ldots, 9\}$ and obtained the following families.

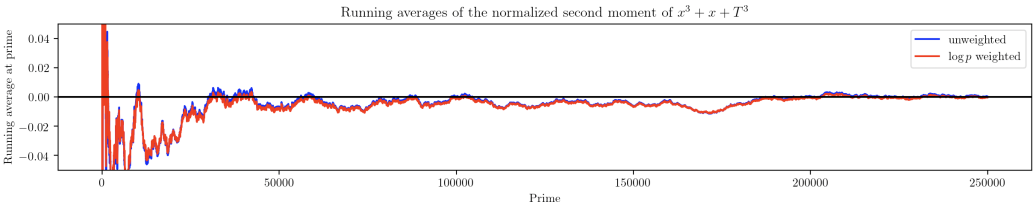


SMALL '24 Generic Random Family 2



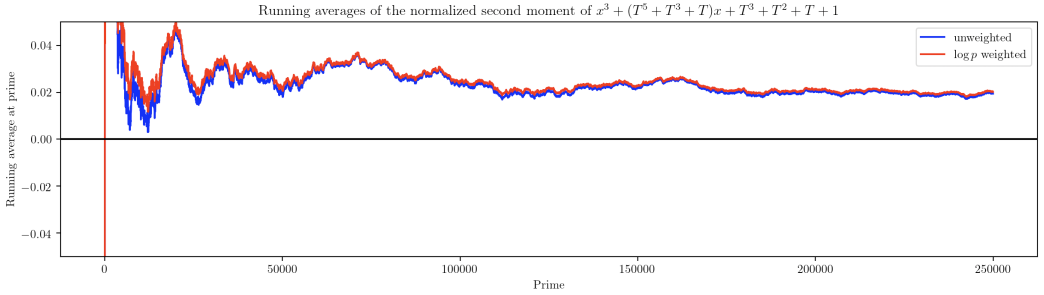
SMALL '23 Family

Recall that the SMALL '23 family $\mathscr{E} : y^2 = x^3 + x + T^3$ has positive bias for all primes $p \equiv 2 \bmod 3$, unknown bias for rest.



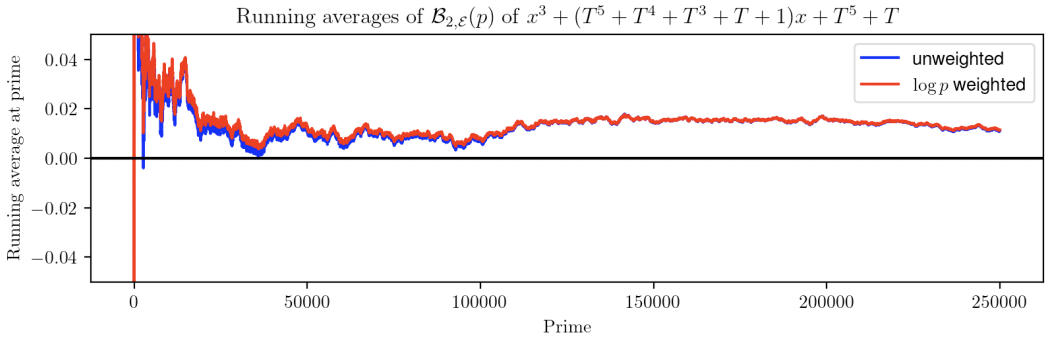
SMALL '24 Potential Positive Bias Family 1: $x^3 + (T^5 + T^3 + T)x + T^3 + T^2 + T + 1$

$$A(T) = T(T^2 - T + 1), \quad B(T) = (T+1)(T^2+1)$$



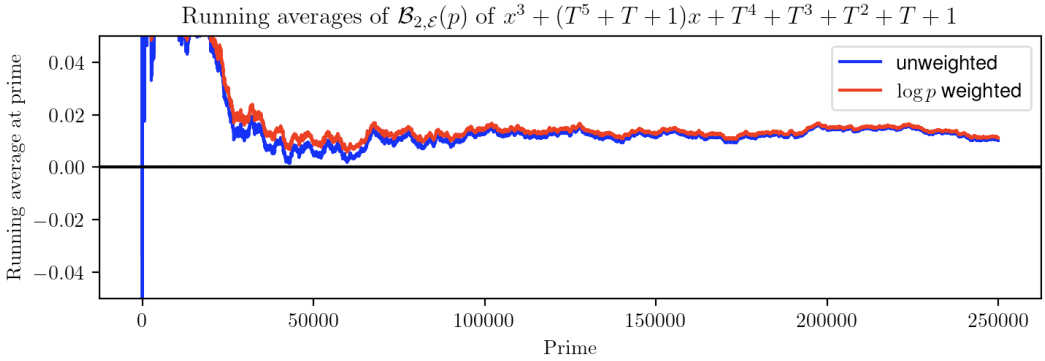
SMALL '24 Potential Positive Bias Family 2: $x^3 + (T^5 + T^4 + T^3 + T + 1)x + T^5 + T$

$$A(T) = (T+1)^2(T^2 - T + 1), \quad B(T) = T(T^4 + 1)$$



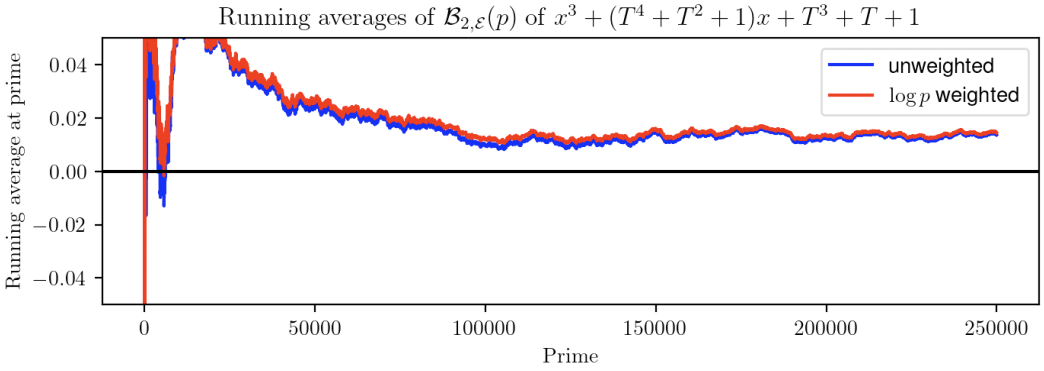
SMALL '24 Potential Positive Bias Family 3: $x^3 + (T^5 + T + 1)x + T^4 + T^3 + T^2 + T + 1$

$$A(T) = (T^2 + T + 1)(T^3 + T^2 - 1), \quad B(T) = (T + 1)(T^2 + 1)$$

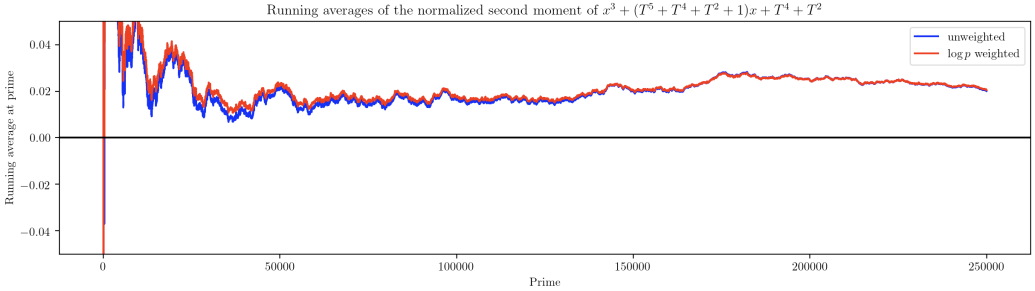


SMALL '24 Potential Positive Bias Family 4: $x^3 + (T^4 + T^2 + 1)x + T^3 + T + 1$

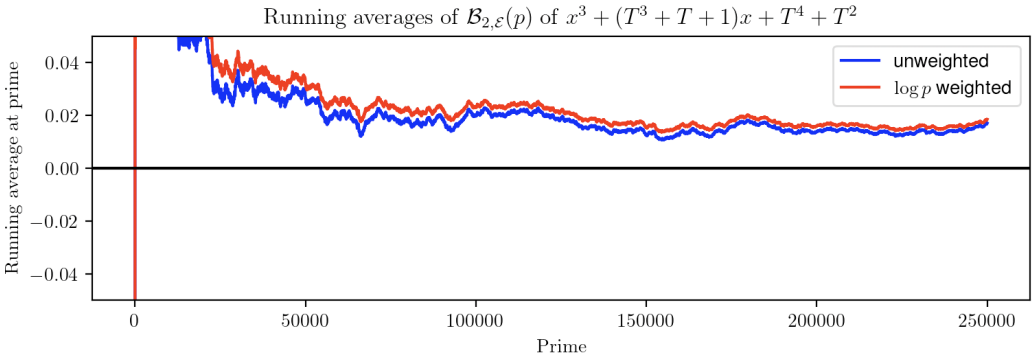
$$A(T) = (T^2 - T + 1)(T^2 + T + 1)$$



SMALL '24 Potential Positive Bias Family 5: $x^3 + (T^5 + T^4 + T^2 + 1)x + T^4 + T^2$

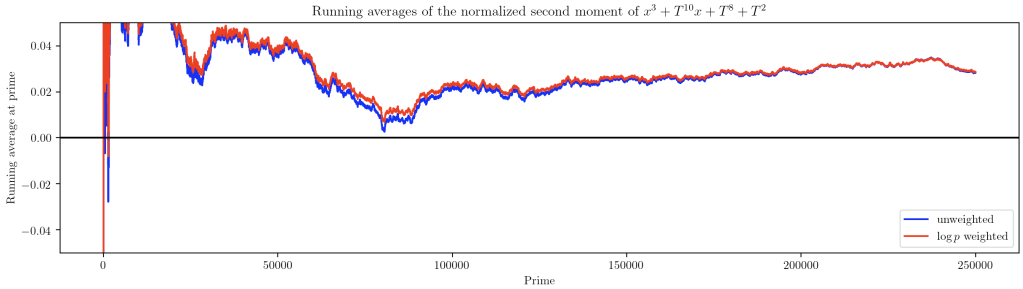


SMALL '24 Potential Positive Bias Family 6: $x^3 + (T^3 + T + 1)x + T^4 + T^2$



SMALL '24 Potential Positive Bias Family 7: $x^3 + T^{10}x + T^8 + T^2$

$$B(T) = T^2(T^4 - T^2 + 1)(T^2 + 1)$$



Assuming there is zero bias, this is a 4.5σ deviation. This happens ~ 3.4 in one million times.

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Motivation

Theorem (Sato-Tate, Taylor et al. 2006 and 2008)

Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication, then $a_p/\sqrt{p}$ converges to a semicircular distribution.

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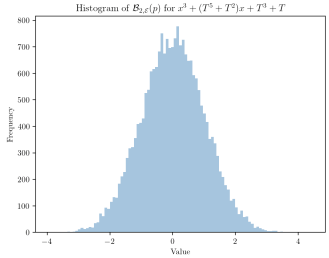
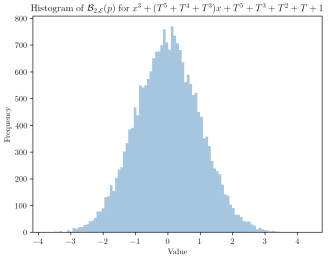
Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication, then $a_p/\sqrt{p}$ converges to a semicircular distribution.

This motivates the exploration of the distribution of the normalized second moment.

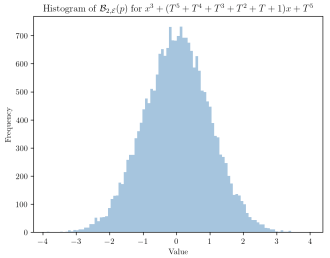
Distribution of the normalized second moment



Variance = 1.0151610262691482



Variance = 1.0046817588446215



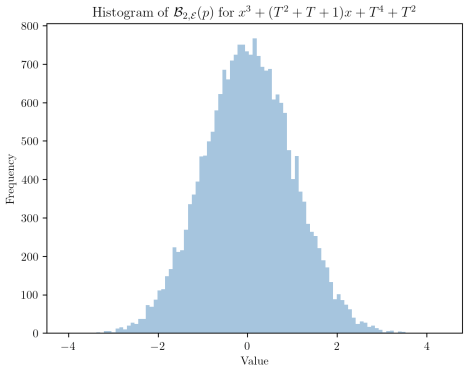
Variance = 0.9891139950619033

Conjecture

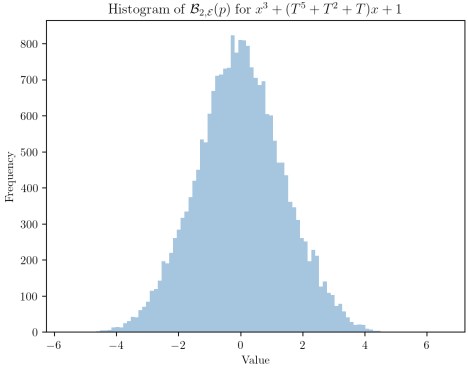
The variance of normalized second moments of a one-parameter family converges to an integer.

Question

What distribution does $\mathscr{B}_{2,\mathscr{E}}(p)$ converge to and what determines the variance?

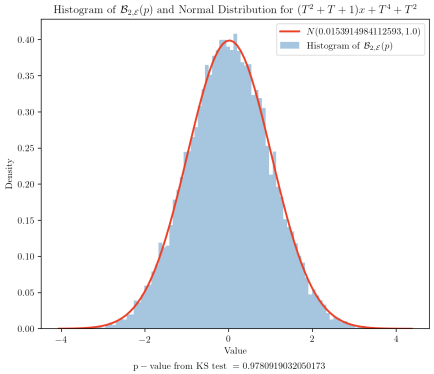


Variance = 0.9941929897237586

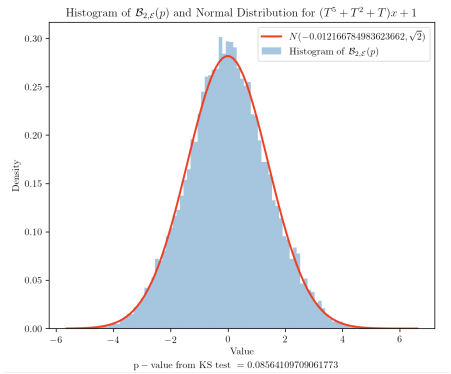


Variance = 1.9905981294487558

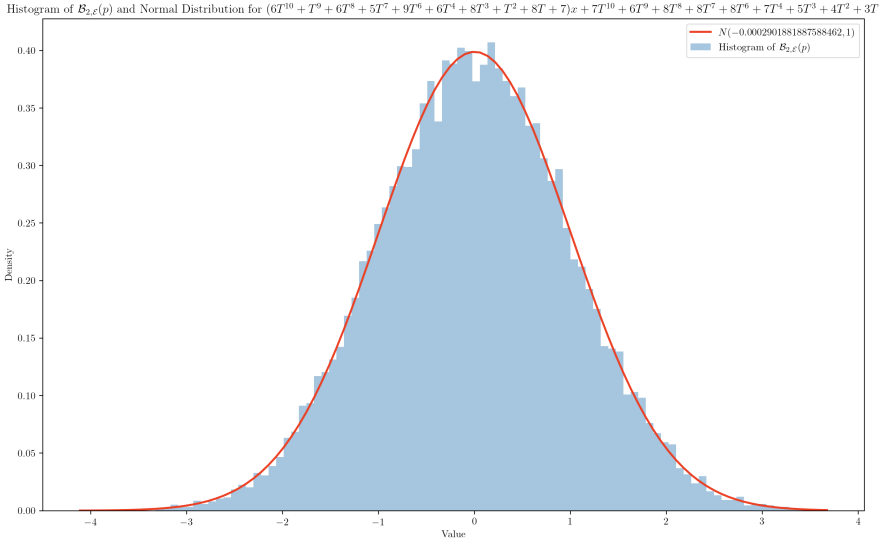
An Open Problem: Gaussian-like Behavior

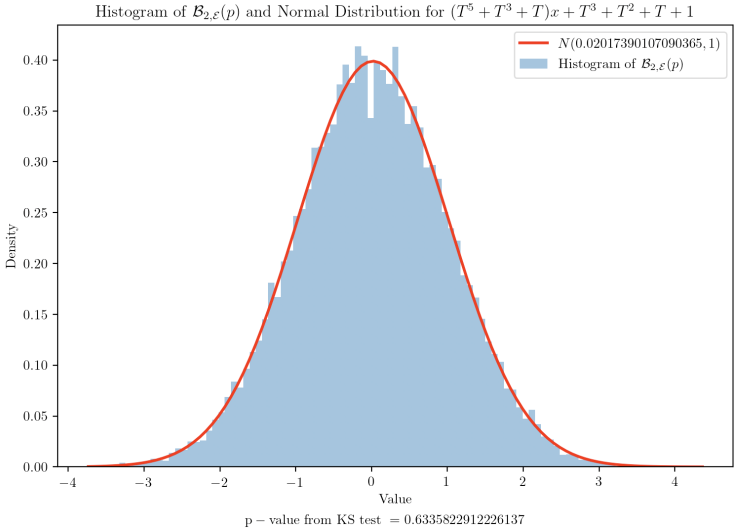


Variance = 0.9941929897237586



Variance = 1.9905981294487558


Histogram of $\mathcal{B}_{2,\mathcal{E}}(p)$ and Normal Distribution for $(6T^{10}+T^9+6T^8+5T^7+9T^6+6T^4+8T^3+T^2+8T+7)x+7T^{10}+6T^9+8T^8+8T^7+8T^6+7T^4+5T^3+4T^2+3T+4$
$N(-0.00029018818875884 62, 1)$
Histogram of $\mathcal{B}_{2,\mathcal{E}}(p)$
Density
Value
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
−4
−3
−2
−1
0
1
2
3
4


Histogram of $\mathcal{B}_{2,\mathcal{E}}(p)$ and Normal Distribution for $(T^5+T^3+T)x+T^3+T^2+T+1$
$N(0.02017390107090365, 1)$
Histogram of $\mathcal{B}_{2,\mathcal{E}}(p)$
Density
Value
p − value from KS test = 0.6335822912226137

1 Introduction

2 Computation of a_p

3 Computational Investigation

4 Normalized Second Moment

5 Algebraic Geometry

Threefolds

We relate the bias in the second moment to point counting on a threefold. For a given one parameter family $y^2 = x^3 + A(T)x + B(T)$, consider the surface

$$y_i^2 z_i = x_i^3 + A(T)x_i z_i^2 + B(T)z_i^3, \qquad \text{for } i = 1, 2.$$

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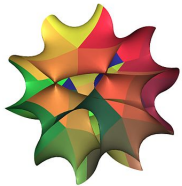
$$y_i^2 z_i = x_i^3 + A(T)x_i z_i^2 + B(T)z_i^3, \quad \text{for } i = 1, 2.$$

To make these calculations possible, we need to be in weighted projective space. If $\deg A \leq 4$ and $\deg B \leq 6$, we let $A(T, U)$ and $B(T, U)$ be the homogenizations of $A(T)$ and $B(T)$ to degree 4 and 6 we consider the threefold

$$V : y_i^2 z_i = x_i^3 + A(T, U)x_i z_i^2 + B(T, U)z_i^3, \quad \text{for } i = 1, 2.$$

Rational Threefold

In general, V is extremely complicated, e.g., a Calabi-Yau manifold:



- However, if for $p > 7$, we can control our elliptic surface mod p, then V becomes rational.
- In some sense this means our singularities are controllable.
- From there one can use known techniques to count points on V mod p.

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We are grateful to Professor Steven J. Miller for suggesting this problem and for his mentorship. We would also like to thank Adam Logan for many helpful conversations.

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