Numerical Investigation of Lower Order Biases in Moment Expansions of One-Parameter Families of Elliptic Curves

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Murmurations in Arithmetic Geometry and Related Topics

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1 Intro to L-Functions

2 Bias Applications

3 Preliminaries

4 Computation of a_p

5 Computational Investigation

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Introduction to L-Functions

Riemann Zeta Function

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) \;=\; \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) \;=\; \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p\left(s,f\right)^{-1}, \quad \mathsf{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f)L(s,f) \;=\; \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

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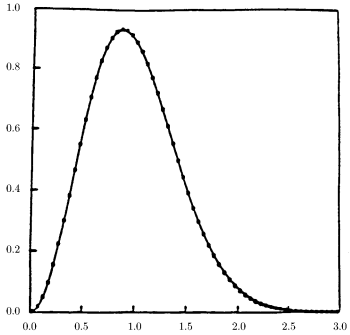
Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko).

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathsf{d}}{\mathsf{d}s}\log\zeta(s) = -\frac{\mathsf{d}}{\mathsf{d}s}\log\prod_p \left(1-p^{-s}\right)^{-1} \\
&= \frac{\mathsf{d}}{\mathsf{d}s}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathsf{Good}(s).
\end{aligned}
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Contour Integration:

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\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \mathsf{vs} \quad \sum_p\log p\int\phi(s)p^{-s}ds.
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Contour Integration (see Fourier Transform arising):

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p\log p\int\phi(s)e^{-\sigma\log p}e^{-it\log p}ds.
$$

Knowledge of Zeros $\Leftrightarrow$ Knowledge of Coefficients.

Explicit Formula: Examples

Dirichlet L-functions: Let h be an even Schwartz function and $L(s,\chi) = \sum_n \chi(n)/n^s$ a Dirichlet L-function from a non-trivial character χ with conductor m and zeros $\rho = \frac{1}{2} + i\gamma_\chi$; if the Generalized Riemann Hypothesis is true then $\gamma \in \mathbb{R}$. Then

$$\begin{aligned}\sum_\rho h\left(\gamma_\rho \frac{\log(m/\pi)}{2\pi}\right) &= \int_{-\infty}^{\infty} h(y)dy \\ -2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{h}\left(\frac{\log p}{\log(m/\pi)}\right) \frac{\chi(p)}{p^{1/2}} & \\ -2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{h}\left(2\frac{\log p}{\log(m/\pi)}\right) \frac{\chi^2(p)}{p} + O\Big(\frac{1}{\log m}\Big). &\end{aligned}$$

Explicit Formula: Examples

Cuspidal Newforms: Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\begin{aligned}\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) &= \widehat{\phi}(0) + \frac{1}{2}\phi(0) - \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} P(f;\phi) \\ &\quad + O\left(\frac{\log\log R}{\log R}\right) \\ P(f;\phi) &= \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.\end{aligned}$$

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbb{R}^{n-1}$.

n-level correlation

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1} - \alpha_{j_2}, \ldots, \alpha_{j_{n-1}} - \alpha_{j_n}\right) \in B, j_i \neq j_k\right\}}{N}$$

(Instead of using a box, can use a smooth test function.)

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbb{R}^{n-1}$.

$\diamond$ Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko).

$\diamond$ 2 and 3-correlations of $\zeta(s)$ (Montgomery, Hejhal).

$\diamond$ n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak).

$\diamond$ n-level correlations for the classical compact groups (Katz-Sarnak).

$\diamond$ Insensitive to any finite set of zeros.

Measures of Spacings: n-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) \quad = \quad \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f\gamma_f^{(j_1)}\right)\cdots\phi_n\left(L_f\gamma_f^{(j_n)}\right)$$

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- ◇ Individual zeros contribute in limit.
- ◇ Most of contribution is from low zeros.
- ◇ Average over similar curves (family).

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Katz-Sarnak Conjecture

For a 'nice' family of L-functions, the n-level density depends only on a symmetry group attached to the family.

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathcal{F}_N|^{-1} \sum_{f \in \mathcal{F}_N} \log C_f$). Hope: ϕ a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\begin{aligned} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) &= \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i \left(\frac{\log C_f}{2\pi} \gamma_E^{(j_i)} \right) \\ &\to \int \cdots \int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x) dx. \end{aligned}$$

Katz-Sarnak Conjecture

As $C_f \to \infty$ the behavior of zeros near $1/2$ agrees with $N \to \infty$ limit of eigenvalues of a classical compact group.

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,\mathrm{SO(even)}}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,\mathrm{SO}}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,\mathrm{SO(odd)}}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1. \end{cases}$$

Correspondences

Similarities between L-Functions and Nuclei:

Zeros	$\longleftrightarrow$	Energy Levels
Schwartz test function	$\longrightarrow$	Neutron
Support of test function	$\longleftrightarrow$	Neutron Energy.

CLT Analogy

Similarities with Central Limit Theorem:
Nice distribution: normalize to mean 0, variance 1.

Higher moments only impact rate of convergence.

For us: first two moments of a_t's give main term.

Care about lower order terms in these / higher moments for rate of convergence.

Previous work looking for closed forms, today will discuss computations.

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Bias Applications

Lower order terms and average rank

$$\frac{1}{N}\sum_{t=N}^{2N}\sum_{\gamma_t}\phi\left(\gamma_t\frac{\log R}{2\pi}\right) = \widehat{\phi}(0)+\phi(0) - \frac{2}{N}\sum_{t=N}^{2N}\sum_{p}\frac{\log p}{\log R}\frac{1}{p}\widehat{\phi}\left(\frac{\log p}{\log R}\right)a_t(p)$$
$$- \frac{2}{N}\sum_{t=N}^{2N}\sum_{p}\frac{\log p}{\log R}\frac{1}{p^2}\widehat{\phi}\left(\frac{2\log p}{\log R}\right)a_t(p)^2 + O\left(\frac{\log\log R}{\log R}\right).$$

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If ϕ non-negative, bound average rank by restricting the sum to be only over zeros at the central point.

Error $O\left(\frac{\log\log R}{\log R}\right)$ from trivial estimation, ignores probable cancellation; expect $O\left(\frac{1}{\log R}\right)$.

For most families $\log R \sim \log N^a$ for some integer a.

Lower order terms and average rank (cont)

The main term of the first and second moments of the $a_t(p)$ give $r\phi(0)$ and $-\frac{1}{2}\phi(0)$.

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Assume second moment of $a_t(p)^2$ is $p^2 - m_{\mathscr{E}}p + O(1)$, $m_{\mathscr{E}} > 0$.

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Assume second moment of $a_t(p)^2$ is $p^2 - m_{\mathscr{E}}p + O(1)$, $m_{\mathscr{E}} > 0$.

Have handled the contribution from p^2; $-m_{\mathscr{E}}p$ contributes

$$\begin{aligned} S_2 &\sim \frac{-2}{N}\sum_p \frac{\log p}{\log R}\widehat{\phi}\left(2\frac{\log p}{\log R}\right)\frac{1}{p^2}\frac{N}{p}(-m_{\mathscr{E}}p) \\ &= \frac{2m_{\mathscr{E}}}{\log R}\sum_p \widehat{\phi}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p^2}. \end{aligned}$$

Thus there is a contribution of size $1/\log R$.

Lower order terms and average rank (cont)

Good choice of (see Appendix A of Iwaniec-Luo-Sarnak (ILS)) is the Fourier pair

$$\phi(x) = \frac{\sin^2(2\pi\frac{\sigma}{2}x)}{(2\pi x)^2}, \quad \widehat{\phi}(u) = \begin{cases} \frac{\sigma-|u|}{4} & \text{if } |u| \le \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Note $\phi(0) = \frac{\sigma^2}{4}$, $\widehat{\phi}(0) = \frac{\sigma}{4} = \frac{\phi(0)}{\sigma}$, and evaluating the prime sum gives

$$S_2 \sim \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2 \log R}\right) \frac{m_{\mathscr{E}}}{\log R}\, \phi(0).$$

Lower order terms and average rank (cont)

Let r_t denote the number of zeros of E_t at the central point (i.e., the analytic rank). Then up to our $O\left(\frac{\log\log R}{\log R}\right)$ errors (which we think should be smaller), we have

$$\begin{aligned}\frac{1}{N}\sum_{t=N}^{2N} r_t\phi(0) &\leq \frac{\phi(0)}{\sigma} + \left(r+\frac{1}{2}\right)\phi(0) + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathscr{E}}}{\log R}\phi(0)\\ \text{Ave Rank}_{[N,2N]}(\mathscr{E}) &\leq \frac{1}{\sigma} + r + \frac{1}{2} + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathscr{E}}}{\log R}.\end{aligned}$$

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$\sigma = 1$, $m_{\mathscr{E}} = 1$: for conductors of size 10^{12}, the average rank is bounded by $1 + r + \frac{1}{2} + .03$ $= r + \frac{1}{2} + 1.03$. This is significantly higher than Fermigier's observed $r + \frac{1}{2} + .40$.

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$\sigma = 2$: lower order correction contributes $.02$ for conductors of size 10^{12}, the average rank bounded by $\frac{1}{2} + r + \frac{1}{2} + .02 = r + \frac{1}{2} + .52$. Now in the ballpark of Fermigier's bound (already there without the potential correction term!).

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Elliptic Curves

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$$E(K) \ := \ \{(x, y) \in K^2 : y^2 = x^3 + Ax + B\} \cup \{\mathcal{O}_E\},$$

with discriminant $\Delta(E) := -16(4A^3 + 27B^2) \neq 0$.

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Theorem (Mordell-Weil 1922)

The set $E(\mathbb{Q})$ is a finitely generated abelian group whose identity element is $\mathcal{O}_E$:

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E(\mathbb{Q})_{\mathrm{tors}},$$

and we call r the rank of $E(\mathbb{Q})$.

Elliptic Curves mod p

From $E/\mathbb{Q}$ we obtain an infinite family of elliptic curves mod p, one for each $p \nmid \Delta(E)$:

- Reduction mod p is a group homomorphism $E(\mathbb{Q}) \longrightarrow E(\mathbb{F}_p)$;
- The expected number of solutions to $E_p := E(\mathbb{F}_p)$ is $p + 1$.

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Theorem (Hasse 1933)

For $a_p := p + 1 - \#E(\mathbb{F}_p)$ we have that $|a_p| \leq 2\sqrt{p}$.

Hasse-Weil L-function

If $p \mid \Delta(E)$, let $\alpha_p := 0, 1$, or -1 based on the type of singularity of E mod p.

Definition

The Hasse-Weil L-function of E is defined by

$$L(s,E) := \prod_{p|\Delta(E)} (1-\alpha_p p^{-s})^{-1} \prod_{p\nmid\Delta(E)} \left(1-a_p p^{-s}+p^{1-2s}\right)^{-1}.$$

The L-function $L(s,E)$ packages the information from the a_p's.

Birch and Swinerton-Dyer

1950s: Computational experiments calculating

$$\pi_E(X) \ := \prod_{p \le X, p \nmid \Delta(E)} \frac{\#E(\mathbb{F}_p)}{p} \ = \prod_{p \le X, p \nmid \Delta(E)} \frac{p+1-a_p}{p}$$

led to the following conjectures.

Conjecture (Birch and Swinnerton-Dyer)

If $E/\mathbb{Q}$ has rank r, then there is some constant C such that $\pi_E(X) \ \sim \ C \log(X)^r$.

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If $E/\mathbb{Q}$ has rank r, then

$$r \ = \ \mathrm{ord}_{s=1} L(s, E).$$

Connecting the two conjectures

If we "formally" evaluate $L(s, E)$ at $s = 1$, then we get

$$L(1, E) \text{ “}=\text{”} \left(\prod_{p|\Delta(E)} \left(1 - \frac{\alpha_p}{p}\right) \prod_{p\nmid\Delta(E)} \frac{p + 1 - a_p}{p} \right)^{-1}.$$

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Theorem (Goldfeld)

Suppose $\pi_E(X) \sim C \log(X)^r$. *Then* $r = \mathrm{ord}_{s=1} L(s, E)$ *and*

$$\lim_{s\to 1} \frac{L(s, E)}{(s-1)^r} = \frac{\sqrt{2}\mathrm{e}^{r\gamma}}{C} \prod_{p|\Delta(E)} (1 - \alpha_p p^{-1})^{-1}.$$

In particular, if $r = 0$, *then* $L(1, E) = \sqrt{2} \left(\prod_{p|\Delta(E)} \left(1 - \frac{\alpha_p}{p}\right) \prod_{p\nmid\Delta(E)} \frac{p+1-a_p}{p} \right)^{-1}$.

One-parameter families

Let $\mathscr{E} \to \mathbb{P}^1$ be a non-split elliptic surface over $\mathbb{Q}$ with

$$\mathscr{E} \ : \ y^2 = x^3 + A(T)x + B(T),$$

where $A(T), B(T) \in \mathbb{Z}[T]$ and $4A(T)^3 + 27B(T)^2 \neq 0$. Almost all specializations $T = t \in \mathbb{Z}$ results in an elliptic curve E_t. This is a *one-parameter family* $\{E_t\}$ of elliptic curves over $\mathbb{Q}$.

- The *rank of a family* is the minimum rank r that appears infinitely often in E_t.

n-th Moments

Definition (n-th Moments of a 1-parameter Family of Elliptic Curves)

Let $a_t(p) := p + 1 - \#E_t(\mathbb{F}_p)$. Then the n-th moment is given by

$$\mathscr{A}_{n,\mathscr{E}}(p) \ := \ \sum_{t=0}^{p-1} a_t(p)^n.$$

Note, we do not normalize by $1/p$.

Moments and the arithmetic properties of elliptic curves

Theorem (Nagao, Rosen-Silverman 1998)

Assuming Tate's conjecture holds for $\mathscr{E}$, we have that

$$\lim_{X\to\infty}\frac{1}{X}\sum_{p\le X}\mathscr{A}_{1,\mathscr{E}}(p)\frac{\log p}{p} \;=\; -\mathrm{rank}\mathscr{E}(\mathbb{Q}(T)).$$

Assuming Tate's conjecture (known for $\mathscr{E}$ if degrees of A, B sufficiently small):

- the rank is determined by the first moment;
- $\sum_{p\le X}\log p \sim X$ so if $\mathscr{A}_{1,\mathscr{E}}(p) = -rp + O(1)$, then $\mathrm{rank}\mathscr{E}(\mathbb{Q}(T)) = r$;
- there is a negative bias in $a_t(p)$.

The Second Moment

The j-invariant of an elliptic surface $\mathscr{E}$ is

$$j(T) \;:=\; 1728\frac{4A(T)^3}{4A(T)^3+27B(T)^2}.$$

Second Moment Asymptotic (Michel, 1995)

For an elliptic surface $\mathscr{E}$ with $j(T)$ non-constant, we have

$$\mathscr{A}_{2,\mathscr{E}}(p) \;=\; p^2+O(p^{3/2}),$$

with lower order terms of size $p^{3/2}, p, p^{1/2}$, and 1.

Bias Conjecture

Question

Do lower order terms in the asymptotic second moment expansion also have a negative bias?

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Conjecture (S.J. Miller '02)

The largest lower order term in the second moment expansion which does not average to 0 is on average negative.

- *Excess rank problem*: disagreement between expected and observed average rank.
- Miller: negative bias in lower order terms increases bound for average rank obtained via 1-level density of zeros near central point.

Strong Bias Conjecture

Conjecture (Strong Bias Conjecture)

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- SMALL '23 disproved the strong bias conjecture with $x^3 + x + T^3$.
- For $p \equiv 2 \bmod 3$, there exists an automorphism $x^3 + x + T^3 \longmapsto x^3 + x + T$.
- The family $x^3 + x + T$ has second moment $p^2 + p$ for $p \equiv 2 \bmod 3$.

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Legendre Symbols

The Legendre symbol is

$$\left(\frac{a}{p}\right) := \begin{cases} 0 \text{ when } a = 0, \\ 1 \text{ when } a \neq 0 \text{ and square}, \\ -1 \text{ otherwise.} \end{cases}$$

Observation

When p is odd and for a given x, the number of solutions to $y^2 = x^3 + ax + b$ mod p is given by

$$1 + \left(\frac{x^3 + ax + b}{p}\right).$$

Legendre Symbols

Lemma

For p odd, then the elliptic curve given by $y^2 = x^3 + ax + b$ satisfies

$$a_p = -\sum_x \left(\frac{x^3 + ax + b}{p}\right).$$

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Consequently, we can calculate the second moment via

$$\mathscr{A}_{2,\mathscr{E}} = \sum_t \sum_{x,y} \left(\frac{x^3 + A(t)x + B(t)}{p}\right)\left(\frac{y^3 + A(t)y + B(t)}{p}\right).$$

In general, explicit formulas are difficult, however, it is possile in some special cases.

Known Expansions

Essentially, the only know non-trivial second moment expansions are as follows:

Family	$A_{1,\mathscr{E}}(p)$	$A_{2,\mathscr{E}}(p)$
$y^2 = x^3 + 2^4(-3)^3(9T+1)^2$	0	$\begin{cases} 2p^2-2p & p \equiv 2 \bmod 3 \\ 0 & p \equiv 1 \bmod 3 \end{cases}$
$y^2 = x^3 \pm 4(4T+2)x$	0	$\begin{cases} 2p^2-2p & p \equiv 1 \bmod 4 \\ 0 & p \equiv 3 \bmod 4 \end{cases}$
$y^2 = x^3 + (T+1)x^2 + Tx$	0	$p^2 - 2p - 1$
$y^2 = x^3 + x^2 + 2T + 1$	0	$p^2 - 2p - 3$
$y^2 = x^3 + Tx^2 + 1$	$-p$	$p^2 - n_{3,2,p}p - 1 + c_{3/2}(p)$
$y^2 = x^3 - T^2x + T^2$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 - T^2x + T^4$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 + Tx^2 - (T+3)x + 1$	$-2c_{p,1;4}p$	$p^2 - 4c_{p,1;6}p - 1$

where $c_{p,a;m} = 1$ if $p \equiv a \bmod m$ and 0 otherwise; $n_{3,2,p}$ is the number of cubes root of 2 mod p; $c_\alpha(p)$ are certain Legendre sums multiplied by p.

Tractable Cases

Closed form expressions via Legendre sums are only tractable in special cases: explicit calculations made by reducing to lower degree polynomials $A(T), B(T)$, and using the identities for linear and quadratic Legendre sums

$$\sum_x \left(\frac{ax+b}{p}\right) = \mathbf{1}_{a=0} p \left(\frac{b}{p}\right)$$

and

$$\sum_x \left(\frac{ax^2+bx+c}{p}\right) = \left(\mathbf{1}_{b^2=4ac} ap - 1\right)\left(\frac{a}{p}\right).$$

Computational Investigation of the Bias Conjecture

- S.J. Miller formulated Bias Conjecture after drawing inspiration from proof of Nagao conjecture, computations for low-degree non-generic families.
- Small degree cases may be misleading: need to study families with higher degree polynomials $A(T), B(T)$ to observe generic families to test Bias Conjecture.
  - Analogously, would never conjecture on primes by just looking at first 10, 20 primes.

Contribution

- We compute a database of a_p for elliptic curves $E_{a,b} : y^2 = x^3 + ax + b$ for all values of $a, b \in \mathbb{F}_p$ for primes p up to $250,000$.

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- This precomputation enables one to quickly study any family of choice (even large ones).
  - for one-parameter family $\mathscr{E}_t$, specialize $t = T$, reduce coeffictions $A(T), B(T)$ $\mathrm{mod} p$, and use our database as lookup table on $(A(T) \bmod p, B(T) \bmod p, p)$ to get corresponding a_p value.

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- Our inital experiments see interesting behavior, and we look for colleagues to try to convert observations to theorems.
  - e.g., study second moments of families with $A(T), B(T)$ of degree ≤ 5.

Constructing Lookup Table

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- A priori, there are p^2-many moment calculations for each p (since there are p^2 elliptic curves over $\mathbb{F}_p$).
- Via automorphisms, to speed up pre-computation, we do better:

$$\#\text{ of } a_p \text{ stored} \;=\; \begin{cases} 4p & p \equiv 1 \bmod 4 \\ 2p & p \equiv 3 \bmod 4 \end{cases} + \begin{cases} 6 & p \equiv 1 \bmod 3 \\ 0 & p \equiv 0, 2 \bmod 3. \end{cases}$$

- We show how to reduce $4p$ appearing above to $2p + 2$.

Automorphisms

Consider the elliptic curve $\mathscr{E}: \ y^2 = x^3 + ax + b$. Letting $y = \ell^3 y'$ and $x = \ell^2 x'$ yields

$$\ell^6 y'^2 \ = \ \ell^6 x'^3 + \ell^2 a x' + b.$$

Dividing by ℓ^6, we conclude

$$\mathscr{E}: \ y'^2 \ = \ x'^3 + \frac{a}{\ell^4} x' + \frac{b}{\ell^6}.$$

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Hence, these two curves are automorphic to each other and have the same number of $\mathbb{F}_p$ points. Thus, it suffices to compute the values of a_p for one a in each quartic residue class. For p odd,

$$\#\text{ of quartic residue classes} = \begin{cases} 5 & p \equiv 1 \pmod 4 \\ 3 & p \equiv 3 \pmod 4. \end{cases}$$

Automorphisms $(a = 0)$

Example

The curves $y^2 = x^3 + 8x + 3$ *and* $y^2 = x^3 + 2x + 9$ *are automorphic mod* 11 *with* $\ell = 3$.

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This suggests that when $a = 0$, we only need to store 6 different a_p values, one for each sextic residue class of b. Hence,

$$\#\text{ of } a_p \text{ we need to store } = \begin{cases} 4p + 6 & p \equiv 1 \pmod 4 \\ 2p + 6 & p \equiv 3 \pmod 4. \end{cases}$$

Automorphisms $(a = 0)$ cont.

When $p \not\equiv 1 \pmod 3$, $x \mapsto x^3$ is an bijection. Hence

$$\sum_x \left(\frac{x^3+b}{p}\right) = \sum_x \left(\frac{x+b}{p}\right) = 0.$$

Therefore, we can compute and store fewer a_p:

$$\#\text{ of } a_p \text{ we need to store } = \begin{cases} 4p & p \equiv 1 \pmod 4 \\ 2p & p \equiv 3 \pmod 4 \end{cases} + \begin{cases} 6 & p \equiv 1 \pmod 3 \\ 0 & p \equiv 0, 2 \pmod 3). \end{cases}$$

Automorphisms $(\sqrt{-1})$

Suppose there is some $i \in \mathbb{F}_p$ such that $i^2 = -1$. Then, put $y = iy'$ and $x = -x$, and thus

$$\begin{aligned} y^2 &= x^3 + ax + b \\ (iy')^2 &= -x^3 - ax + b \\ y'^2 &= x^3 + ax - b. \end{aligned}$$

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Hence, we need to compute a_p for $0 \le b < p/2$. Thus

$$\#\text{ of } a_p \text{ we need to store } = \begin{cases} 2p+2 & p \equiv 1 \pmod 4 \\ 2p & p \equiv 3 \pmod 4 \end{cases} + \begin{cases} 6 & p \equiv 1 \pmod 3 \\ 0 & p \equiv 0, 2 \pmod 3). \end{cases}$$

N.b. this slide has not been implemented due to our code already being sufficiently fast.

Constructing Lookup Table, con.

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  - Once we have found some a with $A(T)a^{-1} \equiv \ell^4 \bmod p$, we compute the corresponding value of $B(T)$ as $B(T)\ell^{-6} \bmod p$.

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  - Once we have found some a with $A(T)a^{-1} \equiv \ell^4 \bmod p$, we compute the corresponding value of $B(T)$ as $B(T)\ell^{-6} \bmod p$.
  - retrieve the a_p value corresponding to (a, b) (i.e., the automorphic curve $y^2 = x^3 + ax + b = x^3 + A(T)\ell^{-4}x + B(T)\ell^{-6}$).

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  - Otherwise, we find sixth power residue class of $B(T)$ by computing $B(T)b^{-1} \bmod p$ for each value of b stored until this is sixth power residue.
  - For such b such that $B(T)b^{-1} \equiv \ell^6 \bmod p$, we retrive a_p value for curve $y^2 = x^3 + b$.

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    - Use trial and error to find $a \in \mathbb{F}_p$ such that $a^2 - x$ is not square (expected number of trials is about 2); no deterministic algorithm known.
    - Then $((a + \sqrt{a^2 - x})^{\frac{p+1}{2}})^2 = x \in \mathbb{F}_p(\sqrt{a^2 - x}) \cong \mathbb{F}_{p^2}$, but $(a + \sqrt{a^2 - x})^{\frac{p+1}{2}} \in \mathbb{F}_p$ since x is a square in $\mathbb{F}_p$.

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Normalized Second Moment

- By Michel's theorem, there are $\alpha(p), \beta(p)$ both $O(1)$ such that

$$\mathscr{A}_{2,\mathscr{E}}(p) \;=\; p^2 + \alpha(p)p^{3/2} + \beta(p)p + O(p^{1/2}).$$

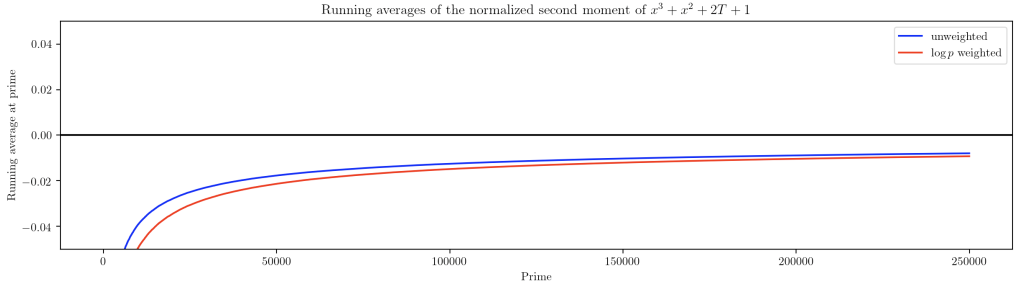
- We compute the *normalized second moment* of $\mathscr{A}_{2,\mathscr{E}}(p)$, which we define to be

$$\mathscr{B}_{2,\mathscr{E}}(p) \;:=\; \frac{\mathscr{A}_{2,\mathscr{E}}(p) - p^2}{p^{3/2}}$$

and graph its running average and $\log p$-weighted running average to find potential positive bias families to investigate further with algebraic geometry.

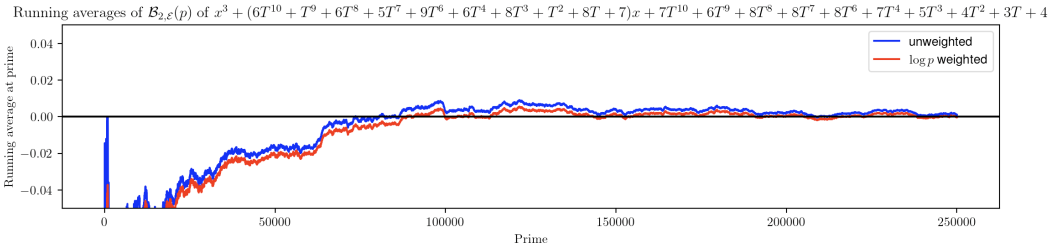
A Family with Known Negative Bias

For $\mathscr{E} : y^2 = x^3 + x^2 + 2T + 1$, we know that $\mathscr{A}_{2,\mathscr{E}}(p) = p^2 - 2p - 3$ has negative bias.

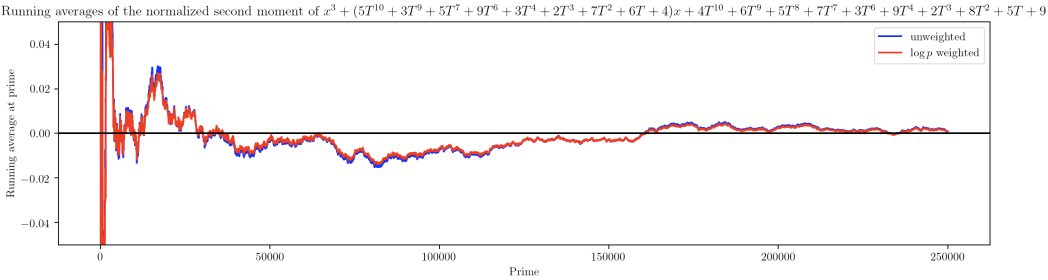


SMALL '24 Generic Random Family 1

We sampled coefficients iid from $\{0, 1, \ldots, 9\}$ and obtained the following families.

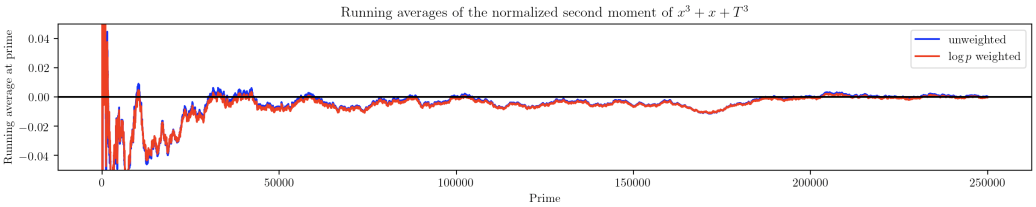


SMALL '24 Generic Random Family 2



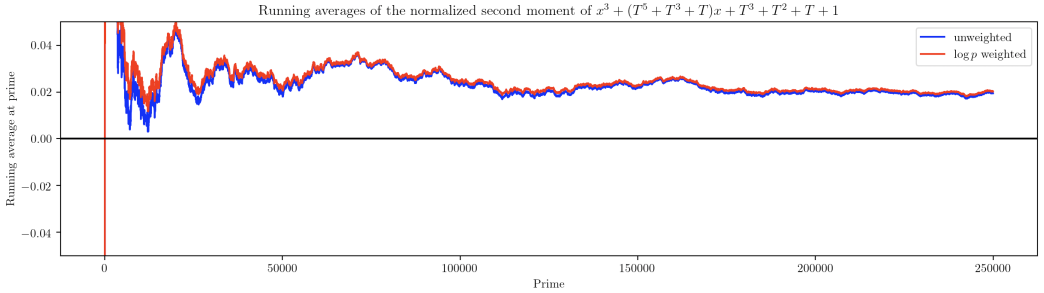
SMALL '23 Family

Recall that the SMALL '23 family $\mathscr{E} : y^2 = x^3 + x + T^3$ has positive bias for all primes $p \equiv 2 \bmod 3$, unknown bias for rest.



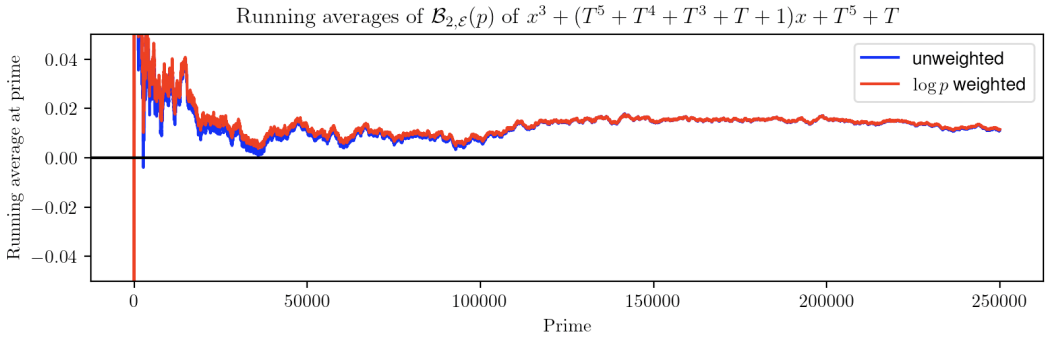
SMALL '24 Potential Positive Bias Family 1: $x^3 + (T^5 + T^3 + T)x + T^3 + T^2 + T + 1$

$$A(T) = T(T^2 - T + 1), \quad B(T) = (T+1)(T^2+1)$$



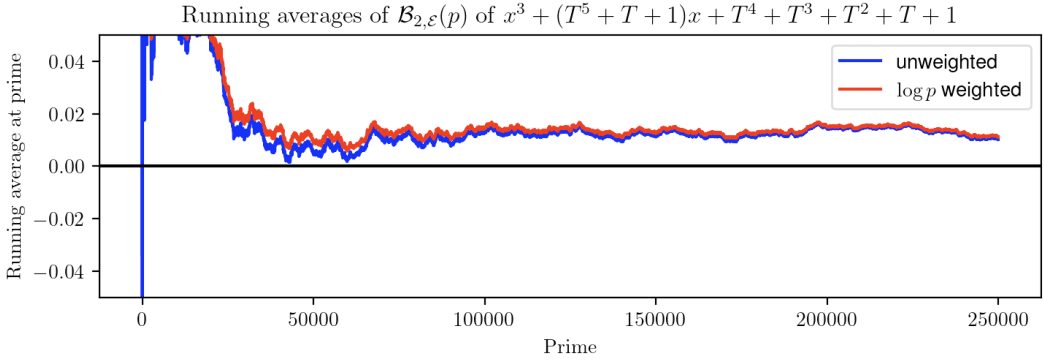
SMALL '24 Potential Positive Bias Family 2: $x^3 + (T^5 + T^4 + T^3 + T + 1)x + T^5 + T$

$$A(T) = (T+1)^2(T^2 - T + 1), \quad B(T) = T(T^4 + 1)$$



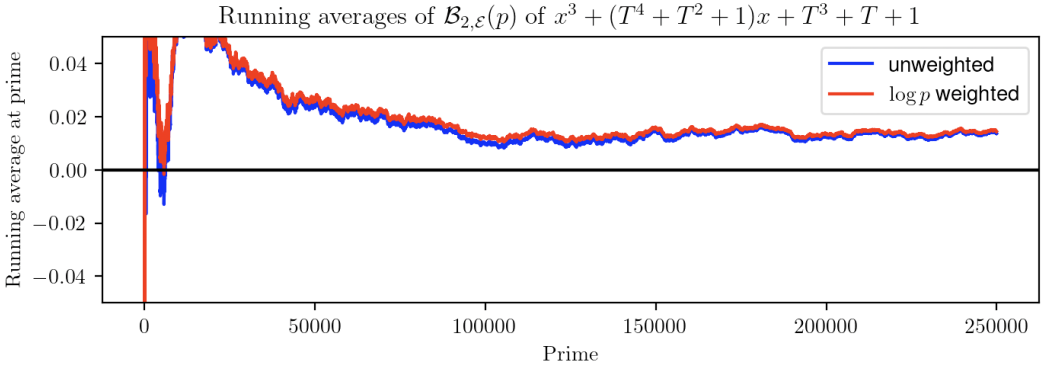
SMALL '24 Potential Positive Bias Family 3: $x^3 + (T^5 + T + 1)x + T^4 + T^3 + T^2 + T + 1$

$$A(T) = (T^2 + T + 1)(T^3 + T^2 - 1), \quad B(T) = (T + 1)(T^2 + 1)$$

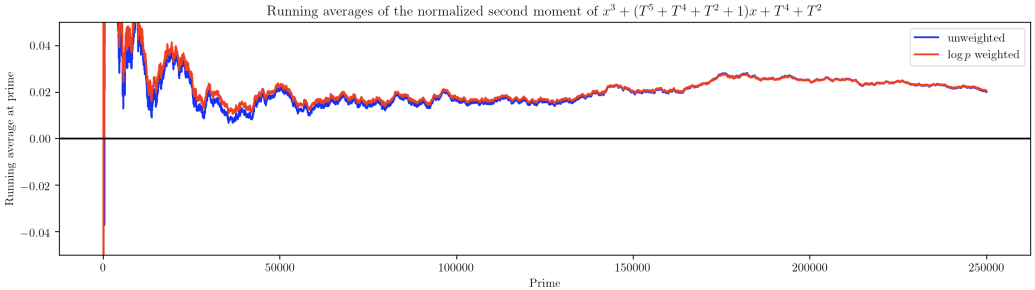


SMALL '24 Potential Positive Bias Family 4: $x^3 + (T^4 + T^2 + 1)x + T^3 + T + 1$

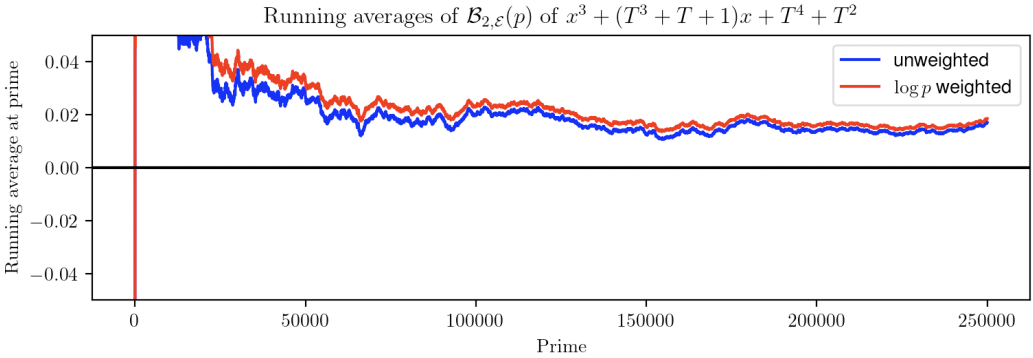
$$A(T) = (T^2 - T + 1)(T^2 + T + 1)$$



SMALL '24 Potential Positive Bias Family 5: $x^3 + (T^5 + T^4 + T^2 + 1)x + T^4 + T^2$

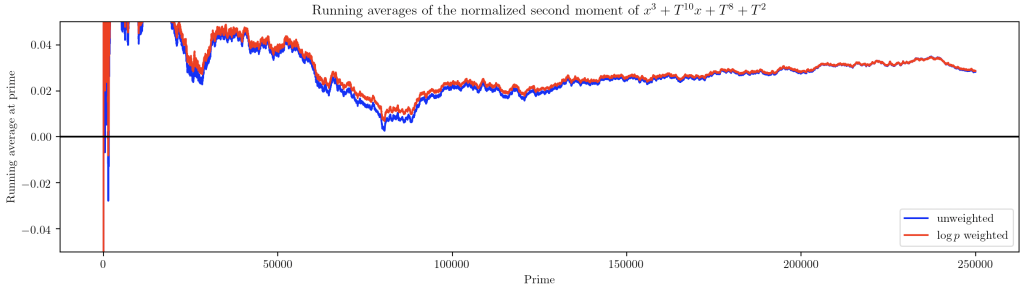


SMALL '24 Potential Positive Bias Family 6: $x^3 + (T^3 + T + 1)x + T^4 + T^2$



SMALL '24 Potential Positive Bias Family 7: $x^3 + T^{10}x + T^8 + T^2$

$$B(T) = T^2(T^4 - T^2 + 1)(T^2 + 1)$$



Assuming there is zero bias, this is a 4.5σ deviation. This happens ~ 3.4 in one million times.

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Motivation

Theorem (Sato-Tate, Taylor et al. 2006 and 2008)

Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication, then $a_p/2\sqrt{p}$ converges to a semicircular distribution.

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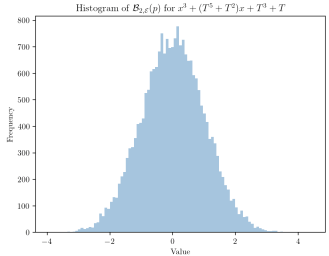
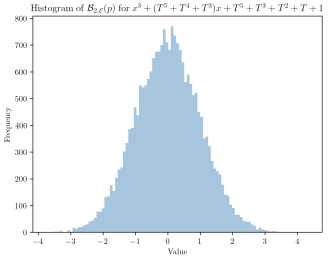
Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication, then $a_p/2\sqrt{p}$ converges to a semicircular distribution.

This motivates the exploration of the distribution of the normalized second moment.

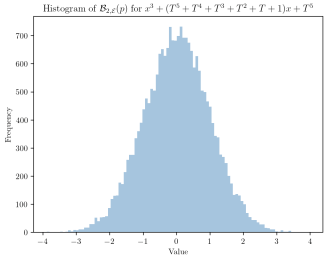
Distribution of the normalized second moment



Variance = 1.0151610262691482



Variance = 1.0046817588446215



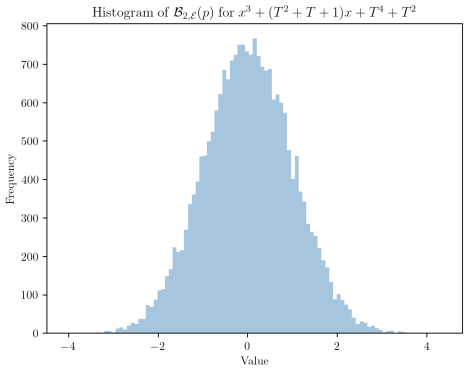
Variance = 0.9891139950619033

Conjecture

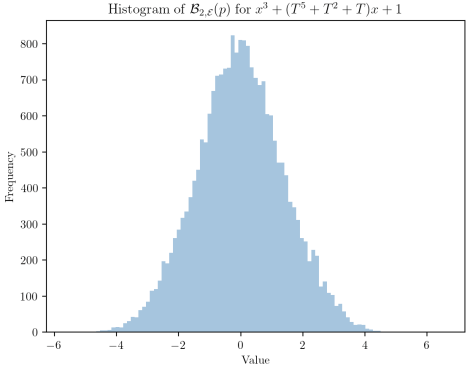
The variance of normalized second moments of a one-parameter family converges to an integer.

Question

What distribution does $\mathscr{B}_{2,\mathscr{E}}(p)$ converge to and what determines the variance?

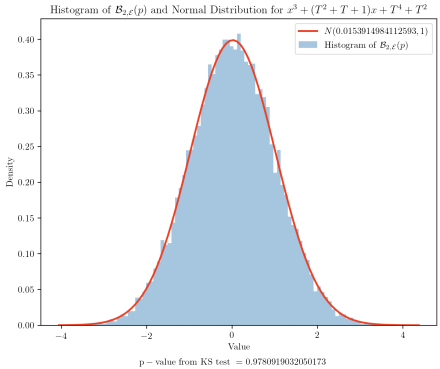


Variance = 0.9941929897237586

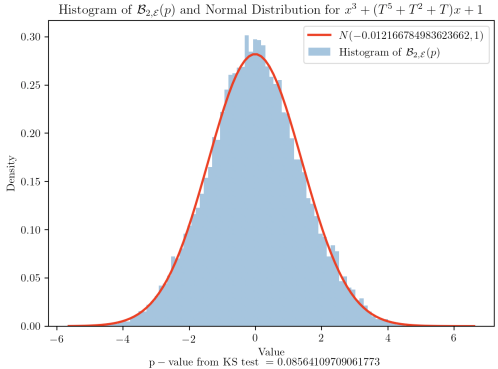


Variance = 1.9905981294487558

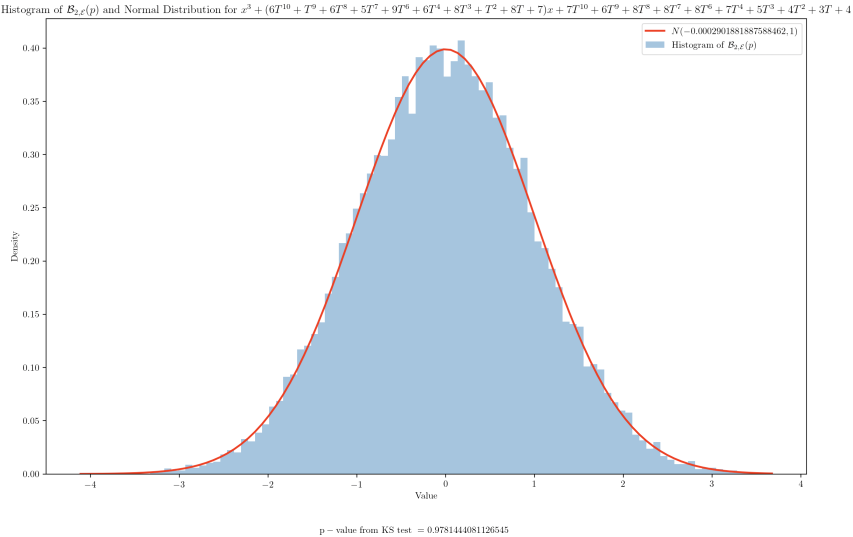
An Open Problem: Gaussian-like Behavior

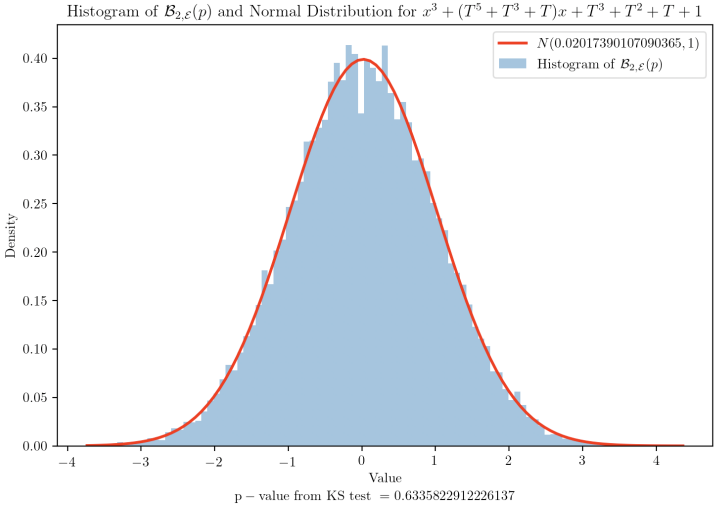


Variance = 0.9941929897237586



Variance = 1.9905981294487558


Histogram of $\mathcal{B}_{2,\mathcal{E}}(p)$ and Normal Distribution for $x^3+(6T^{10}+T^9+6T^8+5T^7+9T^6+6T^4+8T^3+T^2+8T+7)x+7T^{10}+6T^9+8T^8+8T^7+8T^6+7T^4+5T^3+4T^2+3T+4$
$N(-0.00029018818875884 62,1)$
Histogram of $\mathcal{B}_{2,\mathcal{E}}(p)$
Density
Value
p − value from KS test = 0.9781444081126545


Histogram of $\mathcal{B}_{2,\mathcal{E}}(p)$ and Normal Distribution for $x^3 + (T^5 + T^3 + T)x + T^3 + T^2 + T + 1$
$N(0.02017390107090365, 1)$
Histogram of $\mathcal{B}_{2,\mathcal{E}}(p)$
Density
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
−4
−3
−2
−1
0
1
2
3
4
Value
p − value from KS test = 0.6335822912226137

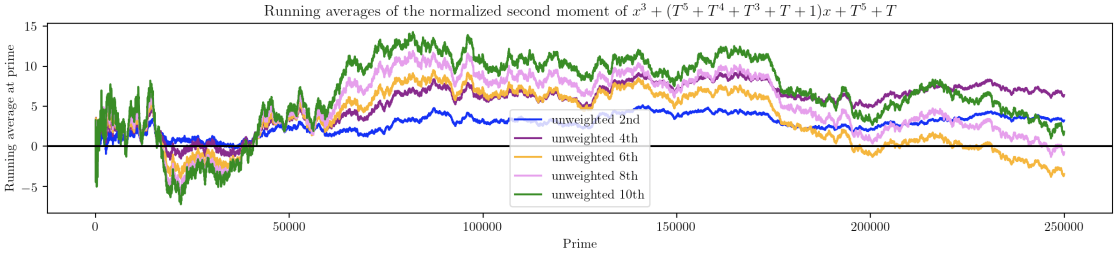
Higher moments

It turns out that

$$\mathscr{A}_{2n,\mathscr{E}}(p) \;=\; C_n p^{n+1} + O(p^{n+1/2}),$$

where C_n denotes the nth Catalan number. If we normalize $\mathscr{A}_{2n,\mathscr{E}}$ via

$$\mathscr{B}_{2n,\mathscr{E}}(p) \;=\; \frac{\mathscr{A}_{2n,\mathscr{E}}(p)}{C_n p^{n+1/2}} - \sqrt{p},$$



We can similarly study the histogram of $\mathscr{B}_{2n,\mathscr{E}}(p)$ values for $n = 2, 3, 4, 5$

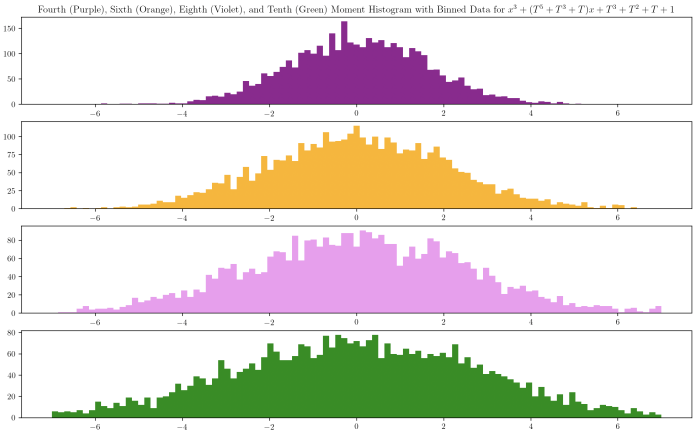


Figure 1: Normalized centered moments split into 100 buckets from -7 to 7

Similarly, we can ask if the variance of these higher moments converges to an integer. The numerical evidence suggests that this is false for moments $2n > 2$.
However, if we define a slightly different normalization for $\mathscr{A}_{2n,\mathscr{E}}$

$$\mathscr{B}'_{2n,\mathscr{E}}(p) \;=\; \frac{\mathscr{A}_{2n,\mathscr{E}}(p)}{p^{n+1/2}} - C_n\sqrt{p},$$

then this effectively multiplies all our normalized values of C_n, and thus multiplies the variance of C_n^2. Here, for $p \leq 250,000$ single data points can change the variance by .1 or more, and thus we do not have enough data to conjecture or reject that the variance converges to an integer.

1 Intro to L-Functions

2 Bias Applications

3 Preliminaries

4 Computation of a_p

5 Computational Investigation

6 Normalized Second Moment

7 Algebraic Geometry

Threefolds

We relate the bias in the second moment to point counting on a threefold. For a given one parameter family $y^2 = x^3 + A(T)x + B(T)$, consider the surface

$$y_i^2 z_i = x_i^3 + A(T)x_i z_i^2 + B(T)z_i^3, \qquad \text{for } i = 1, 2.$$

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$$y_i^2 z_i = x_i^3 + A(T)x_i z_i^2 + B(T)z_i^3, \quad \text{for } i = 1, 2.$$

To make these calculations possible, we need to be in weighted projective space. If $\deg A \leq 4$ and $\deg B \leq 6$, we let $A(T, U)$ and $B(T, U)$ be the homogenizations of $A(T)$ and $B(T)$ to degree 4 and 6 we consider the threefold

$$V : y_i^2 z_i = x_i^3 + A(T, U)x_i z_i^2 + B(T, U)z_i^3, \quad \text{for } i = 1, 2.$$

Rational Threefold

In general, V is extremely complicated, e.g., a Calabi-Yau manifold:

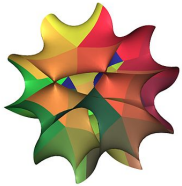

Figure from Andrew J. Hanson's "A construction for computer visualization of certain complex curves," Notices of the Amer. Math. Soc. 41 (9): 1156-1163, (November/December 1994)

- However, if for $p > 7$, we can control our elliptic surface mod p, then V becomes rational.
- In some sense this means our singularities are controllable.
- From there one can use known techniques to count points on V mod p.

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