Benford's Law and Dependent Random Variables

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Joint Meetings of the AMS / MAA, Boston
January 5, 2011

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- products of independent random variables
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Question: Which systems lead to Benford behavior?

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Our motivating example (Lemons 1986):
Decomposition of a conserved quantity as a model for particle decay.

A Model for Decomposition of Conserved Quantities

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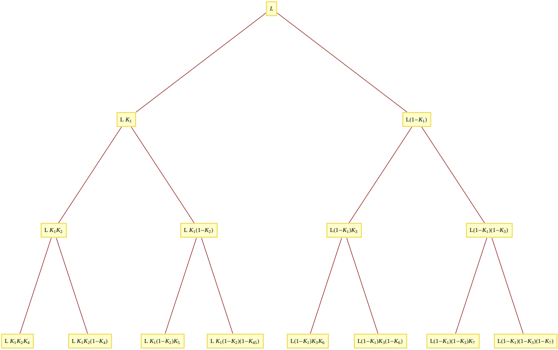

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For $s \in [1, 10)$, let

$$\varphi_s(u) \ := \ \begin{cases} 1 & \text{if } u\text{'s significand is at most } s \\ 0 & \text{otherwise.} \end{cases}$$

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Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed modulo 1, where $y_i = \log_B x_i$.

The Mellin Transform

Let $f(x)$ be a continuous real-valued function on $[0, \infty)$. Define its Mellin transform, $(\mathcal{M}f)(s)$, by

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- With a logarithmic change of variables, we can translate between Mellin and Fourier transforms.

Products of Independent Random Variables

Jang, Kang, Kruckman, Kudo and Miller (2009)

$\Xi_1, \dots, \Xi_N$ independent variables with densities f_{Ξ_m} and

$$\lim_{N\to\infty} \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} \prod_{m=1}^{N} (\mathcal{M}f_{\Xi_m})\left(1 - \frac{2\pi i\ell}{\log B}\right) \;=\; 0 \;\; (C1).$$

Then as $\Xi_1 \cdots \Xi_N$ converges to Benford's law.

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Then as $\Xi_1 \cdots \Xi_N$ converges to Benford's law.

- (C1) is quite weak, met by most distributions.
- **Corollary:** For $\Xi_1, \dots, \Xi_N$ uniformly distributed on $(0,1)$ and $N \geq 4$,
  $$|\varphi_s(u) - \log_{10} s| \leq \left(\frac{1}{2.9^N} + \frac{\zeta(N)-1}{2.7^N}\right) 2\log_{10} s.$$

Limiting Behavior of Decompostions

Theorem (Decomposition model as above)

Fix a continuous density f on $[0,1]$ st $f(x)$, $f(1-x)$ satisfy (C1), set

$$P_N(s) \ := \ \frac{\sum_{i=1}^{2^N} \varphi_s(X_i)}{2^N},$$

the fraction of pieces with significand less than or equal to s. Then

1. $\lim_{N\to\infty} \mathbb{E}[P_N(s)] = \log_{10} s.$
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- Amalgamation of processes converges to Benford.
- May consider a single process (if number stages tend to infinity).

Convergence of Amalgamated Lengths to Benford

$$\mathbb{E}[P_N(s)] = \mathbb{E}\left[\frac{\sum_{i=1}^{2^N} \varphi_s(X_i)}{2^N}\right] = \frac{1}{2^N}\sum_{i=1}^{2^N} \mathbb{E}[\varphi_s(X_i)].$$

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We can apply JKKKM's theorem, as the Mellin transform at $1 - \frac{2\pi i \ell}{\log 10}$ is strictly less than 1.

For specific choices of f, can obtain precise bounds on the error. Ex: $\mathbb{E}[\varphi_s(X_i)] - \log_{10} s \ll \frac{1}{2.9^N}$.

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- Study integral as a function of the significand of first $M+1$ variables, which we can bound by JKKKM.
- Remaining task is to count for each of the 2^N choices of i and for $1 \leq n \leq N$, how many choices of X_j have n factors not in common with X_i.
- Resulting sum bounds variance above and goes to 0, thus $\lim_{N\to\infty} \mathrm{Var}\left(P_N(s)\right) = 0$.

Summary of Results

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  - Pieces must sum to original stick length, thus they are dependent.
  - Allowable densities obey weak condition C1.
- Expected lengths of pieces from amalgamation of processes converges to Benford distribution.
  - Key observation: dependencies exist among piece lengths, not densities.
- Variance in piece length distribution goes to zero for a single process. Calculation complicated by dependencies:
  - Study expectation cross terms as integrals over their independent factors.

Thanks to:

- AMS / MAA
- Williams College SMALL 2011
- National Science Foundation
- Thealexa Becker, Professor Steven J. Miller, Ryan Ronan, Professor Frederick W. Strauch