Moment Formulas for Ensembles of Classical Compact Groups

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Introduction

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \dots$.

Question: What rules govern the spacings between the t_i?

Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w $n^k\alpha$ mod 1.
- Spacings b/w Zeros of L-functions.

Classical
Random Matrix Theory

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei (Uranium: 200$+$ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

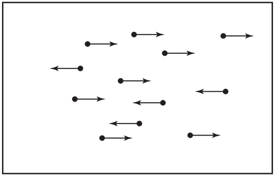
$$H\psi_n \;=\; E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Origins of Random Matrix Theory



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

To each A, attach a probability measure:

$$\begin{aligned}
\mu_{A,N}(x) &= \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right) \\
\int_a^b \mu_{A,N}(x)dx &= \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N} \\
\mathrm{k^{th}\ moment} &= \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^kN^{\frac{k}{2}+1}} = \frac{\mathrm{Trace}(A^k)}{2^kN^{\frac{k}{2}+1}}.
\end{aligned}$$

L-functions

Riemann Zeta Function

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) \;=\; \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) \;=\; \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

General L-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) = \Lambda_\infty(s,f) L(s,f) = \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (GRH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Properties of zeros of L-functions

- infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from L-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.

Katz-Sarnak Density Conjectures

Measures of Spacings: n-Level Density and Families

Let g_i be even Schwartz functions whose Fourier Transform is compactly supported, $L(s, f)$ an L-function with zeros $\frac{1}{2} + \mathrm{i}\gamma_f$ and conductor Q_f:

$$D_{n,f}(g) = \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\gamma_{f,j_1}\frac{\log Q_f}{2\pi}\right)\cdots g_n\left(\gamma_{f,j_n}\frac{\log Q_f}{2\pi}\right)$$

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- Properties of *n*-level density:
  - ◇ Individual zeros contribute in limit
  - ◇ Most of contribution is from low zeros
  - ◇ Average over similar *L*-functions (family)

n-Level Density

n-level density: $\mathcal{F} = \cup \mathcal{F}_N$ a family of L-functions ordered by conductors, ϕ_k an even Schwartz function: $D_{n,\mathcal{F}}(\phi) =$

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \phi_1\left(\frac{\log Q_f}{2\pi}\gamma_{j_1;f}\right)\cdots\phi_n\left(\frac{\log Q_f}{2\pi}\gamma_{j_n;f}\right)$$

As $N \to \infty$, n-level density converges to

$$\int \phi(\overrightarrow{x})\rho_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{x})d\overrightarrow{x} = \int \widehat{\phi}(\overrightarrow{u})\widehat{\rho}_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{u})d\overrightarrow{u}.$$

Conjecture (Katz-Sarnak)

(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

Correspondences

Similarities between L-Functions and Nuclei:

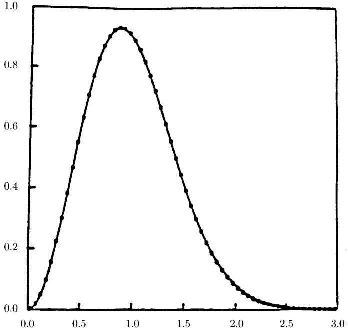
$$\text{Zeros} \quad \longleftrightarrow \quad \text{Energy Levels}$$

$$\text{Schwartz test function} \quad \longrightarrow \quad \text{Neutron}$$

$$\text{Support of test function} \quad \longleftrightarrow \quad \text{Neutron Energy.}$$

Theory: New Combinatorial Vantage

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko) versus RMT prediction.

Key Kloosterman-Bessel integral (cuspidal newforms)

Ramanujan sum:

$$R(n,q) \;=\; \sum_{a \bmod q}^{*} e(an/q) \;=\; \sum_{d|(n,q)} \mu(q/d)d,$$

where $*$ restricts the summation to be over all a relatively prime to q.

Theorem (ILS)

Let Ψ be an even Schwartz function with $\operatorname{supp}(\widehat{\Psi}) \subset (-2,2)$. Then

$$\sum_{m \le N^{\epsilon}} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y)\widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}y}{\log R}$$
$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty}\Psi(x)\frac{\sin 2\pi x}{2\pi x}\mathrm{d}x - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k\log\log kN}{\log kN}\right),$$

where $R = k^2N$ and φ is Euler's totient function.

2-Level Density (cuspidal newforms)

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)J_{k-1}\left(\frac{4\pi\sqrt{m^2x_1x_2N}}{c}\right)\frac{dx_1dx_2}{\sqrt{x_1x_2}}$$

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Change of variables and Jacobean:

$$\begin{aligned} u_2 &= x_1 x_2 & \qquad x_2 &= \frac{u_2}{u_1} \\ u_1 &= x_1 & \qquad x_1 &= u_1 \end{aligned}$$

$$\left|\frac{\partial x}{\partial u}\right| = \begin{vmatrix} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} \end{vmatrix} = \frac{1}{u_1}.$$

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$$\int\int \widehat{\phi}\left(\frac{\log u_1}{\log R}\right)\widehat{\phi}\left(\frac{\log\left(\frac{u_2}{u_1}\right)}{\log R}\right) J_{k-1}\left(\frac{4\pi\sqrt{m^2 u_2 N}}{c}\right)\frac{du_1 du_2}{u_1\sqrt{u_2}}.$$

2-Level Density (cuspidal newforms)

Changing variables, u_1-integral is

$$\int_{w_1=\frac{\log u_2}{\log R}-\sigma}^{\sigma} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right)dw_1.$$

Support conditions imply

$$\Psi_2\left(\frac{\log u_2}{\log R}\right) \;=\; \int_{w_1=-\infty}^{\infty} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right)dw_1.$$

Substituting gives

$$\int_{u_2=0}^{\infty} J_{k-1}\left(\frac{4\pi\sqrt{m^2u_2N}}{c}\right)\Psi_2\left(\frac{\log u_2}{\log R}\right)\frac{du_2}{\sqrt{u_2}}$$

n-Level Density: Katz-Sarnak Determinant Expansions

- $\mathrm{U(N)}$, $\mathrm{U}_k(N)$: $\det\left(K_0(x_j,x_k)\right)_{1\le j,k\le n}$
- $\mathrm{USp(N)}$: $\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\le n}$
- $\mathrm{SO(even)}$: $\det\left(K_1(x_j,x_k)\right)_{1\le j,k\le n}$
- $\mathrm{SO(odd)}$: $\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\le n} + \sum_{\nu=1}^{n}\delta(x_\nu)\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\ne\nu\le n}$

where

$$K_\epsilon(x,y) = \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)} + \epsilon\frac{\sin\left(\pi(x+y)\right)}{\pi(x+y)}.$$

Preliminaries

Manipulating determinant expansions leads to analysis of

$$K(y_1,\dots,y_n) = \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\dots+\lambda_m=n \\ \lambda_j\ge 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!} \sum_{\epsilon_1,\dots,\epsilon_n=\pm1} \prod_{\ell=1}^{m} \chi_{\left\{\left|\sum_{j=1}^{n}\eta(\ell,j)\epsilon_j y_j\right|\le 1\right\}},$$

where

$$\eta(\ell,j) = \begin{cases} +1 & \text{if } j \le \sum_{k=1}^{\ell}\lambda_k \\ -1 & \text{if } j > \sum_{k=1}^{\ell}\lambda_k. \end{cases}$$

Small support: $\text{supp}(\widehat{\phi}) \subset (-1/n, 1/n)$

If $\text{supp}(\phi) \subseteq [-\frac{1}{n}, \frac{1}{n}]$, then each $|y_j| \le 1/n$ and

$$\prod_{\ell=1}^{m} \chi_{\{|\sum_{j=1}^{n} \eta(\ell,j)\epsilon_j y_j| \le 1\}}$$

is always 1. Therefore the sum is

$$2^n \cdot \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\cdots+\lambda_m=n \\ \lambda_j \ge 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1! \cdots \lambda_m!} \;=\; 0.$$

New Results: Hughes-Miller (Orthogonal), Iyer-Miller (Unitary, Symplectic): supp$(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

If supp$(\widehat{\phi}) \subseteq [-\frac{1}{n-1}, \frac{1}{n-1}]$, then $\left|\sum_{j=1}^{n} \eta(\ell, j)\epsilon_j y_j\right| > 1$ only when all $\eta(\ell, j)\epsilon_j y_j$ have the same sign.

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Contributes

$$\sum_{m=1}^{n} \sum_{\substack{\lambda_1+\cdots+\lambda_m=n \\ \lambda_j \geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1! \cdots \lambda_m!}(2^n - 2m) \ = \ 2(-1)^n.$$

New Result: Iyer-Miller: Large support: $\text{supp}(\widehat{\phi}) \subseteq [-\frac{1}{n-2}, \frac{1}{n-2}]$

If $\text{supp}(\widehat{\phi}) \subseteq [-\frac{1}{n-2}, \frac{1}{n-2}]$, two new complications:

1. $\eta(\ell, j)\epsilon_j y_j$ need not have same sign (at most one can differ);
2. more than one term in product can be zero (for fixed m, λ_j, ϵ_j).

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Solution: Double count (gives a contribution of $4(-1)^n$), and subtract a correcting term ρ_j.

New Result: Iyer-Miller: Large support: $\text{supp}(\widehat{\phi}) \subseteq [-\frac{1}{n-2}, \frac{1}{n-2}]$

Solution: Work with cumulants instead of moments:

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Solution: Work with cumulants instead of moments:

$$\begin{aligned} C_n^{SO}(\phi) &= 4(-1)^{n-1}\left(\int_{-\infty}^{\infty}\phi(x)^n\cdot\frac{\sin(2\pi x)}{2\pi x}dx - \frac{1}{2}\phi(0)^n\right) \\ &\quad + \frac{1}{4}\left(2n(-1)^n + \sum_{j=1}^{n}\rho_j\right) \\ &\quad \times\left(\phi^n(0) - \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi^{n-1}(x_1)\hat{\phi}(x_2)(1+|x_2|)\right. \\ &\quad \left.\times e^{2\pi i x_1|x_2|}\frac{\sin(2\pi x_1)}{2\pi x_1}dx_1dx_2\right). \end{aligned}$$

Conclusion

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1. Difficulty in comparison with classical RMT is that instead of having an n-dimensional integral of $\phi_1(x_1)\cdots\phi_n(x_n)$ we have a 1-dimensional integral of a new test function. This leads to harder combinatorics but allows us to appeal to the result from ILS.

2. Solve combinatorics by using cumulants; support restrictions translate to which terms can contribute.