Unitary similarity to a complex symmetric matrix and its extension to orthogonal symmetric Lie algebra

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Abstract

Some characterizations of a square complex matrix being unitarily similar to a symmetric matrix are given. Our approach uses singular value decomposition. A result of Vermeer is extended in the context of orthogonal symmetric Lie algebra of the compact type.

Outline

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- Introduction

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- Some Characterizations

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Theorem (Vermeer 2008)

Let $A \in \mathbb{C}_{n\times n}$. The following statements are equivalent.

1. A is unitarily similar to a complex symmetric matrix.
2. UAU^* is symmetric for some symmetric unitary matrix U.
3. $A = SU$ for some symmetric matrix S and symmetric unitary matrix U.
4. $VAV^* = A^T$ for some symmetric unitary matrix V.

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Let $A \in \mathbb{C}_{n\times n}$ and let $A = V\Sigma U^*$ be a *singular value decomposition* (SVD) of A, where $\Sigma = diag(\sigma_1, \dots, \sigma_n)$ and $\sigma_1 \geq \cdots \geq \sigma_n$ are the singular values of A. Notice that $U, V \in U(n)$ are not uniquely determined. However, if the singular values are distinct, then U and V are respectively unique up to the post-multiplication of a diagonal unitary matrix. Clearly $A^*AU = U\Sigma^2$ and $AA^*V = V\Sigma^2$.

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Lemma(Autonne Decomposition)

If $A \in \mathbb{C}_{n\times n}$ is symmetric, then there is $U \in U(n)$ such that $A = U\Sigma U^T$. In particular, if A is symmetric unitary, then $A = UU^T$ for some $U \in U(n)$.

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Let $A \in \mathbb{C}_{n\times n}$ and let $\Sigma = \sigma_1 I_{n_1} \oplus \cdots \oplus \sigma_m I_{n_m}$, where $\sigma_1 > \cdots > \sigma_m$ are the distinct singular values of A with multiplicities $n_1, \ldots, n_m$ respectively. The following statements are equivalent.

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1. *A is unitarily similar to a complex symmetric matrix.*
2. *There are $X, Y \in U(n)$ such that X^*Y is symmetric for some SVD $A = Y\Sigma X^*$.*
3. *For any SVD $A = V\Sigma U^*$, the matrix $(UQ)^*VQ'$ is symmetric for some $Q := Q_1 \oplus \cdots \oplus Q_m \in U(n)$ and $Q' := Q_1 \oplus \cdots \oplus Q_{m-1} \oplus Q'_m \in U(n)$, with $Q_i \in U(n_i)$, $i = 1, \ldots, m-1$, and $Q_m, Q'_m \in U(n_m)$. If, in addition, $\sigma_m > 0$, then $Q'_m = Q_m$, i.e., $Q' = Q$.*

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(1) $\Rightarrow$ (2). Suppose that, $A = W^*SW$ with $W \in U(n)$. Then $S = Z^T \Sigma Z$ for some $Z \in U(n)$ by the above Lemma. Thus $A = (W^* Z^T)\Sigma(ZW)$ is a SVD of A with $(ZW)(W^* Z^T) = ZZ^T$ symmetric.

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(2) $\Rightarrow$ (3).Since $U, X \in U(n)$ and the columns of U and X are eigenvectors of A^*A corresponding to the eigenvalues σ_i^2's, we have $X = UQ$ for some $Q := Q_1 \oplus \cdots \oplus Q_m$, where $Q_j \in U(n_j)$, $j = 1, \ldots, m$. Similarly $Y = VQ'$ for some $Q' := Q_1' \oplus \cdots \oplus Q_m'$, where $Q_j' \in U(n_j)$, $j = 1, \ldots, m$. Thus $(UQ)^*(VQ') = X^*Y$ is symmetric. Notice that

$$A = Y\Sigma X^* = VQ'\Sigma(UQ)^* = V\Sigma Q' Q^* U^*$$

so that $\Sigma Q' Q^* = V^* A U = \Sigma$. Since $\sigma_{m-1} > 0$, $Q_i' Q_i^* = I_{n_i}$, i.e., $Q_i' = Q_i$, for $i = 1, \ldots, m-1$. In addition, if $\sigma_m > 0$, then $Q_m' = Q_m$ as well, i.e., $Q' = Q$.

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is a SVD of A. Since $(UQ)^*VQ'$ is symmetric unitary, $(UQ)^*VQ' = ZZ^T$ for some $Z \in U(n)$ by Autonne Decomposition Lemma. Set $W := Z^*(UQ)^* \in U(n)$ so that $(UQ)^* = ZW$ and $VQ' = (UQ)ZZ^T = W^*Z^T$. Thus $A = W^*(Z^T\Sigma Z)W$, i.e., A is unitarily similar to the complex symmetric matrix $Z^T\Sigma Z$.

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Corollary

Let $A \in \mathbb{C}_{n\times n}$ be nonsingular and $A = V\Sigma U^*$ a SVD of A. Then A is unitarily similar to a complex symmetric matrix if and only if $A_d := V(\Sigma - dI)U^*$ is unitarily similar to a complex symmetric matrix for any $d \leq \sigma_m$, where σ_m is the smallest singular value of A.

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- Let $\mathfrak{u}$ be a compact semisimple Lie algebra and θ an involutive automorphism of $\mathfrak{u}$. Then $\mathfrak{u} = \mathfrak{k}_0 \oplus \mathfrak{p}_*$, where $\mathfrak{k}_0$ and $\mathfrak{p}_*$ are the $+1$ and -1 eigenspaces of θ respectively. It is easy to see that

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subset \mathfrak{k}_0, \quad [\mathfrak{k}_0, \mathfrak{p}_*] \subset \mathfrak{p}_*, \quad [\mathfrak{p}_*, \mathfrak{p}_*] \subset \mathfrak{k}_0.$$

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- Let (U, K) be a pair associated with $(\mathfrak{u}, \theta)$. The pair $(\mathfrak{u}, \theta)$ or (U, K) is said to be an orthogonal symmetric pair of the compact type, and U/K is a Riemannian locally symmetric space.
- The extension (also denoted by θ) of θ to $\mathfrak{g} := \mathfrak{u}_{\mathbb{C}} = \mathfrak{u} \oplus i\mathfrak{u}$ defined by

$$\theta(X + iY) = \theta(X) + i\theta(Y), \qquad \text{for any } X, Y \in \mathfrak{u}$$

is a (complex) involutive automorphism of $\mathfrak{g}$.

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Let $(\mathfrak{u}, \theta)$ be an orthogonal symmetric Lie algebra of the compact type. Let $\mathfrak{u} = \mathfrak{k}_0 \oplus \mathfrak{p}_*$ be the direct decomposition of $\mathfrak{u}$ into ± 1-eigenspaces of θ. Let (U, K) be a pair associated with $(\mathfrak{u}, \theta)$. The following statements are equivalent for any $A \in \mathfrak{g} := \mathfrak{u} \oplus i\mathfrak{u}$.

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1. $Ad_U(u)A$ is invariant under $-\theta$ (θ, respectively) for some $u \in U$.

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1. $Ad_U(u)A$ is invariant under $-\theta$ (θ, respectively) for some $u\in U$.
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3. $Ad_U(u)A = -\theta(A)$ ($\theta(A)$, respectively) for some $u \in \exp \mathfrak{p}_*$.

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The previous Theorem is essentially corresponding to the orthogonal symmetric pair $(\mathfrak{u}, \theta)$ with $\mathfrak{u} = \mathfrak{su}(n)$ and $\theta(X) = \bar{X}$, for which we have that $\mathfrak{k}_0 = \mathfrak{so}(n)$, $\mathfrak{p}_*$ consists of all purely imaginary symmetric traceless matrices, $U = SU(n)$, $K = SO(n)$, and $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) = \mathfrak{su}(n) \oplus i\mathfrak{su}(n)$. The extension of θ to $\mathfrak{g}$ is defined by

$$\theta(X+iY) = \theta X + i\theta Y = -(\bar{X})^* - i(\bar{Y})^* = -(X+iY)^T, X, Y \in \mathfrak{su}(n).$$

Thus $\theta(A) = -A^T$ for all $A \in \mathfrak{sl}_n(\mathbb{C})$ so that taking transpose on $\mathfrak{sl}_n(\mathbb{C})$ amounts to $-\theta$ in the previous Theorem . Thus the $-\theta$-invariant set Ω in $\mathfrak{sl}_n(\mathbb{C})$ consists of all traceless symmetric matrices.

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We remark that if we consider θ in the previous Theorem instead of $-\theta$, where $\theta(A) = -A^T$ for all $A \in \mathfrak{sl}_n(\mathbb{C})$, we then have the following result, with an appropriate translation.

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3. *$VAV^* = 2(trA/n)I - A^T$ for some symmetric unitary matrix V.*

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