Characterizing Homeomorphisms on Cantor Sets by Orbit Structure

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Outline

1. Main Theorem
2. Today's Case
3. Tracing the Proof
4. Results

Definitions for our Motivation

Definition

Let $T : X \to X$. The *orbit spectrum* of T is the sequence

$$\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \cdots)$$

of cardinals, where ν is the number of $\mathbb{N}$-orbits, ζ is the number of $\mathbb{Z}$-orbits and σ_n is the number of $n - cycles$.

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We say that a subset $N \subset \mathbb{N}$ is *finitely generated* if there is a collection $\{n_1, \cdots, n_k\} \subset N$ such that for every $j \in N$, there exists $i \leq k$ with $n_i | j$.

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The orbit spectrum $\sigma(T)$ is *finitely based* if $\{n \in \omega : \sigma_n \neq 0\}$ is finitely generated.

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Given $T : X \to X$ with orbit spectrum $\sigma(T)$, we say that j is a *stray period* if $\sigma_j \neq 0$ and for all $k \in \mathbb{N}$, $\sigma_{jk} < \mathfrak{c}$. We say that a periodic orbit $\mathcal{O}(x) = \{x_0, \cdots, x_{n-1}\}$ is a *stray orbit* if n is a stray period.

Cantor Set Result

Theorem

Let X be a Cantor set. There is a homeomorphism $T : X \to X$ with $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \ldots)$ if and only if one of the following holds:

1. $\zeta = \mathfrak{c}$,
2. $1 \leq \zeta < \mathfrak{c}$, $\{n : \sigma_n = \mathfrak{c}\}$ *is infinite, and* $\sum\{\sigma_n : n \text{ is a stray period}\} \leq \zeta$, *or*
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Goal

We'll trace the ideas of the proof of the following theorem.

Theorem

If X is a compact metric space and $T : X \to X$ is a homeomorphism with $\zeta(T) = 0$ and $\sigma(T)$ not finitely based, then X has a non-degenerate connected component.

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Theorem

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Suppose:

1. X is compact metric,
2. $T : X \to X$ is a homeomorphism,
3. $\zeta(T) = 0$ and
4. $\sigma(T)$ is not finitely based.

Since $\sigma(T)$ is not finitely based:

Y

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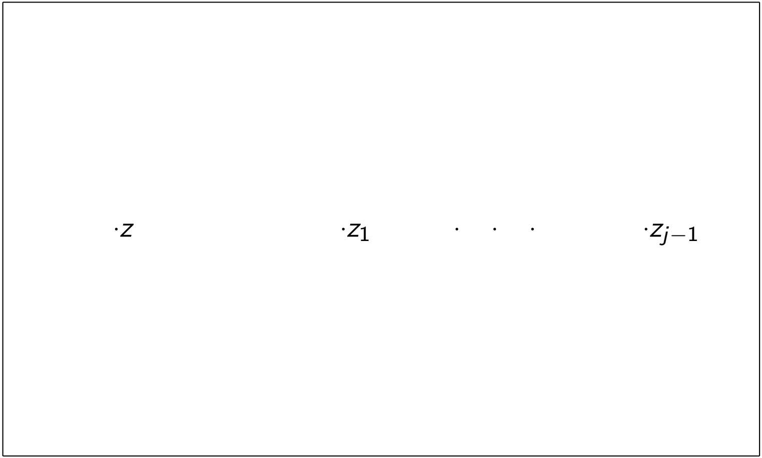

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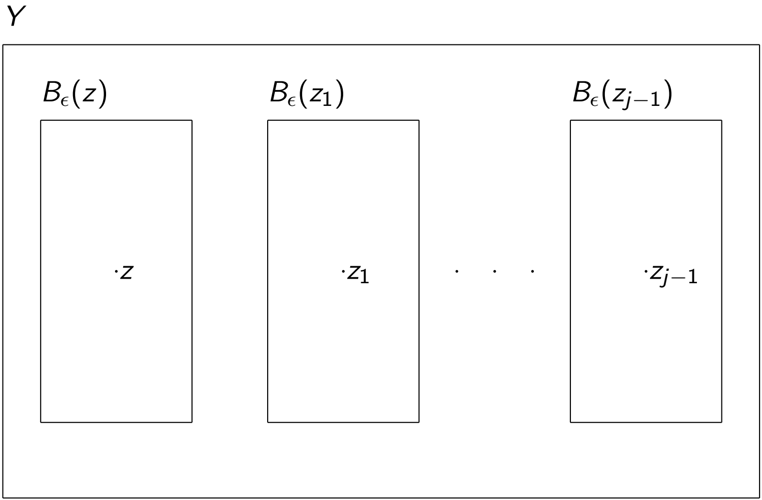

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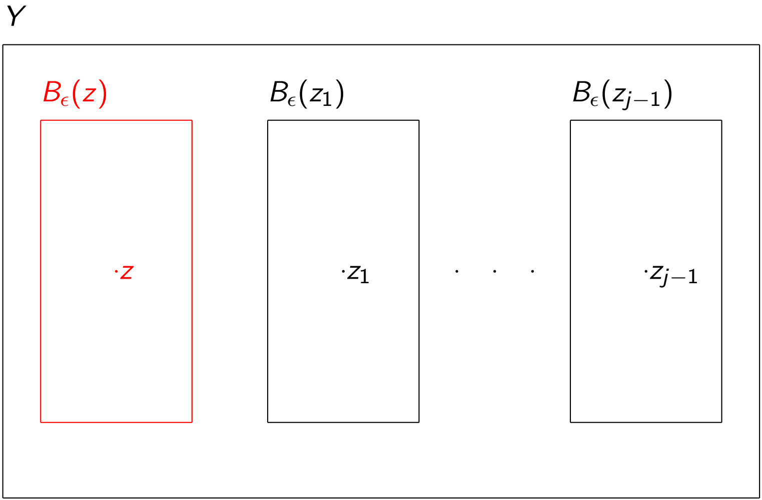

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Our goal is to find a nondegenerate connected subset inside of Y. To do so, we will use the following characterization of components in compact metric spaces.

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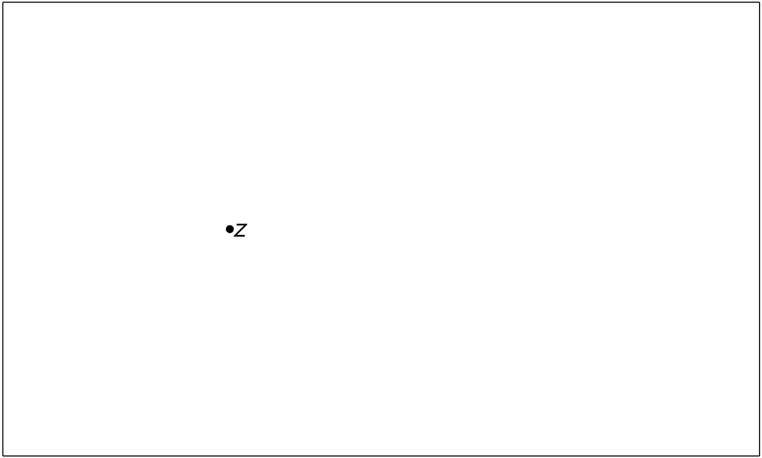
Theorem

Let X be a compact metric space, and let $x, y \in X$ with $x \neq y$. Suppose that for every $n \in \mathbb{N}$ there is a chain, $\mathcal{C} = \{C_1, C_2 \ldots C_m\}$, of nonempty open sets in X with diameter less than $\frac{1}{n}$ such that $x \in C_1$ and $y \in C_m$. Then there is a non-degenerate component of X containing x and y.

Very not to scale:

$\overline{B_\epsilon(z)}$



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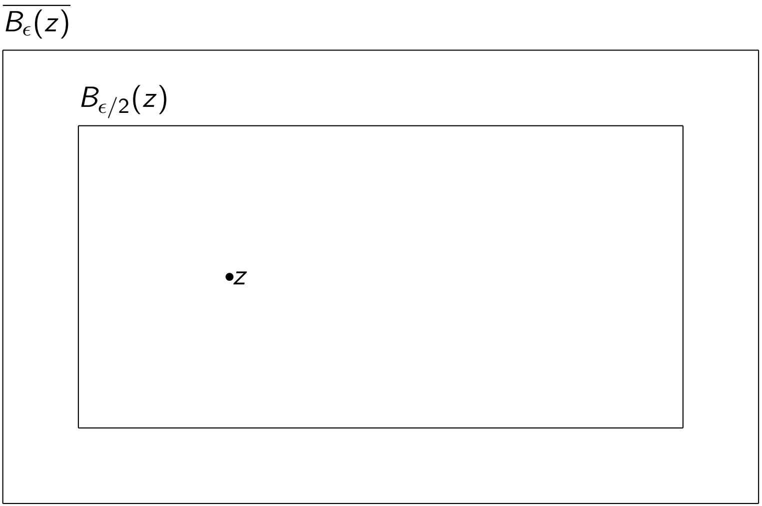

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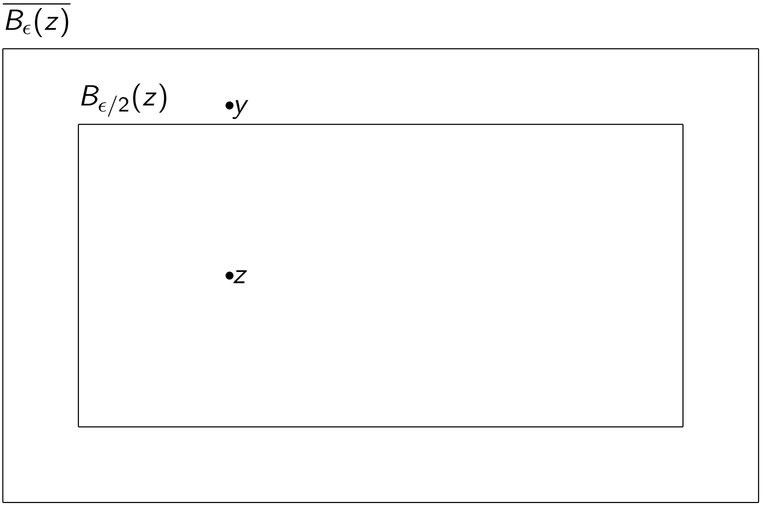

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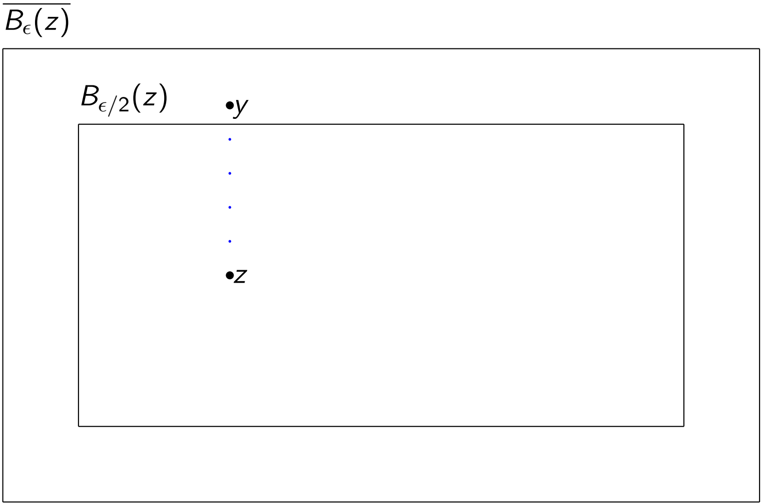

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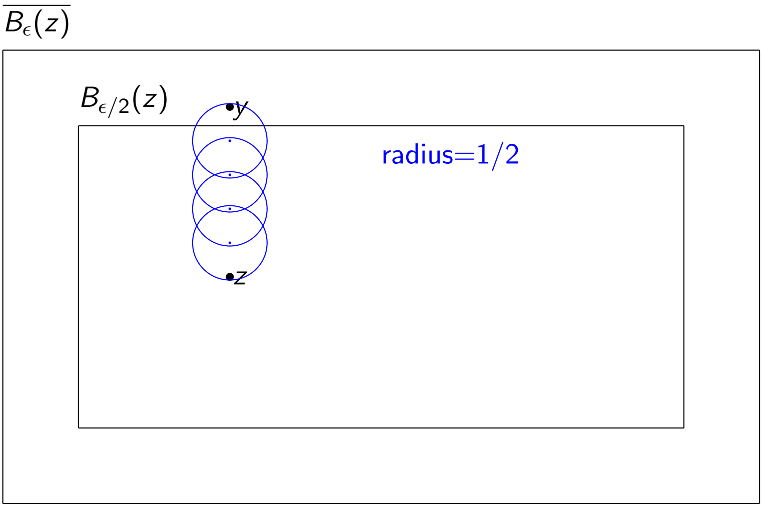

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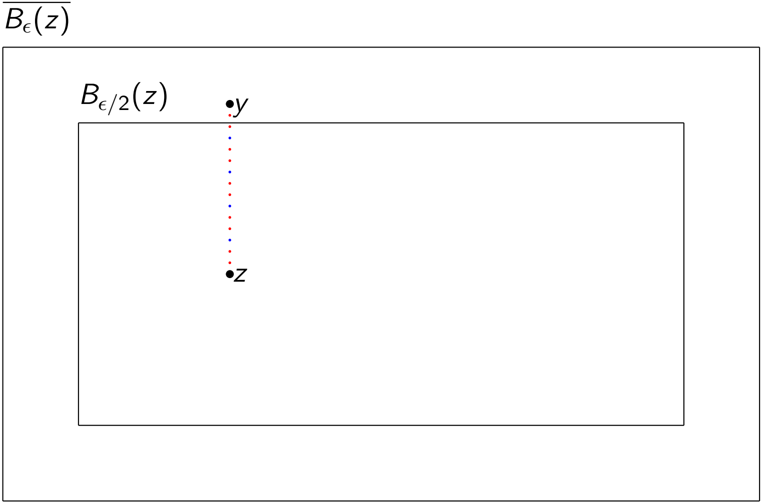

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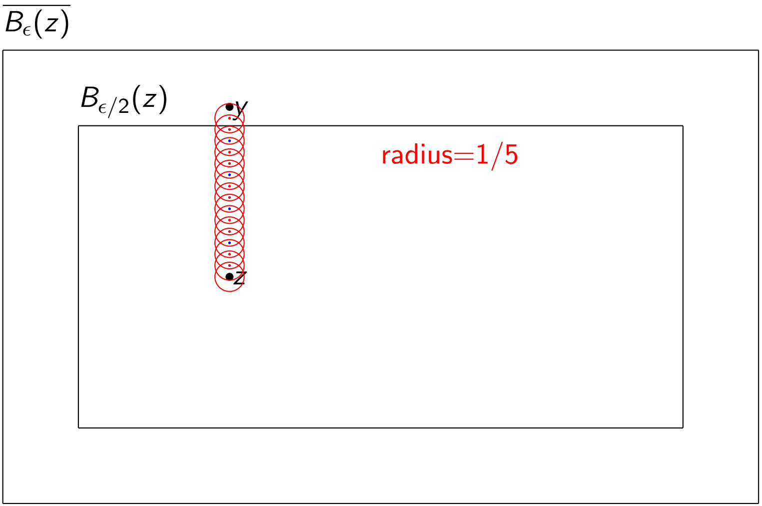

ALSO since $\sigma(T)$ is not finitely based:

Given $\delta > 0$

1. there exists a closed subspace $Y(\delta)$ of Y such that every point in $Y(\delta)$ is within δ of a point of period j. (and the orbit of z is still in the interior of $Y(\delta)$ relative to $Y(\delta)$ and relative to Y)

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We now wish to demonstrate that given $n \in \mathbb{N}$, there exists an n-chain from z to a point in $\overline{B_\epsilon(z)} \setminus B_{\epsilon/2}(z)$, so we let $n \in \mathbb{N}$ be chosen. (We may as well assume $1/2n < \epsilon$)

Choosing δ

By the uniform continuity of T we can choose $0 < \delta < 1/2n$ small enough that if $x \in B_\delta(z)$ and both

1. $per(x) > j$ and
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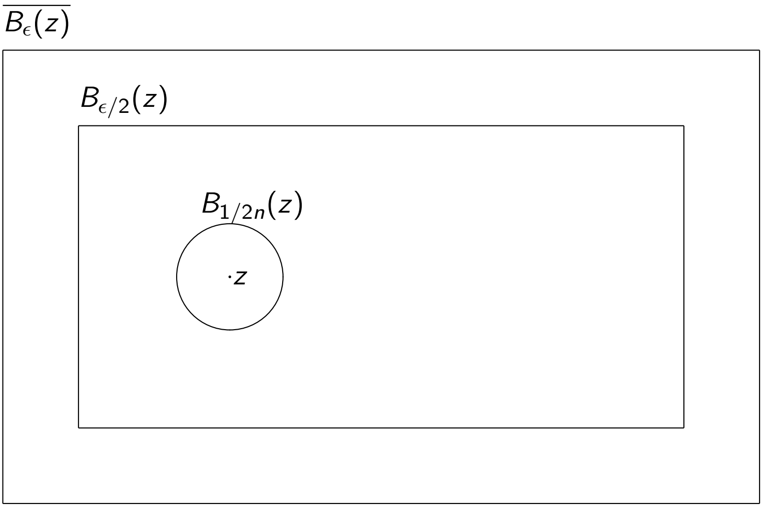
2. there exists a least n such that $T^{nj}(x) \notin B_\delta(z)$,
3. $d(T^{(n-1)j}x, T^{nj}x) \leq d(T^{(n-1)j}x, z) + d(z, T^{nj}x) < \delta + 1/2n < 1/n$.

We will now assume we are looking inside $Y(\delta)$.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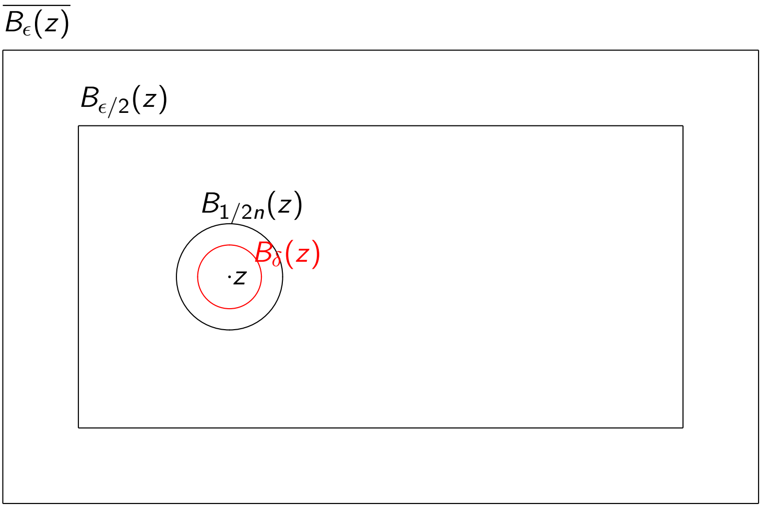
$\cdot z$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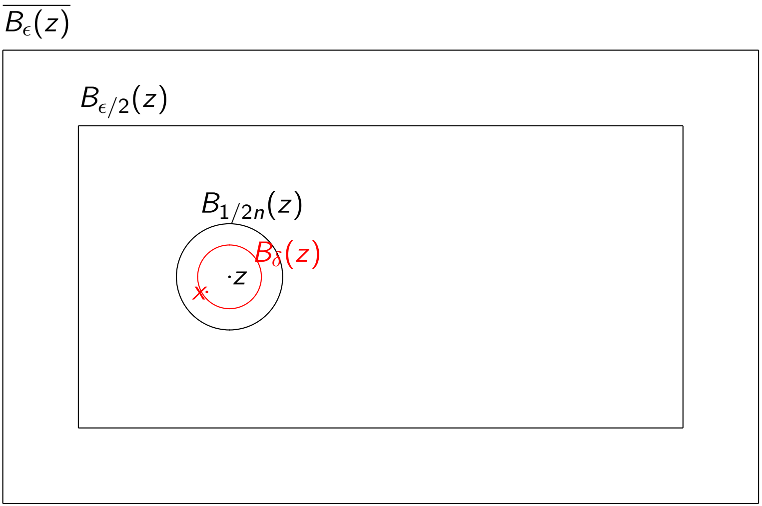
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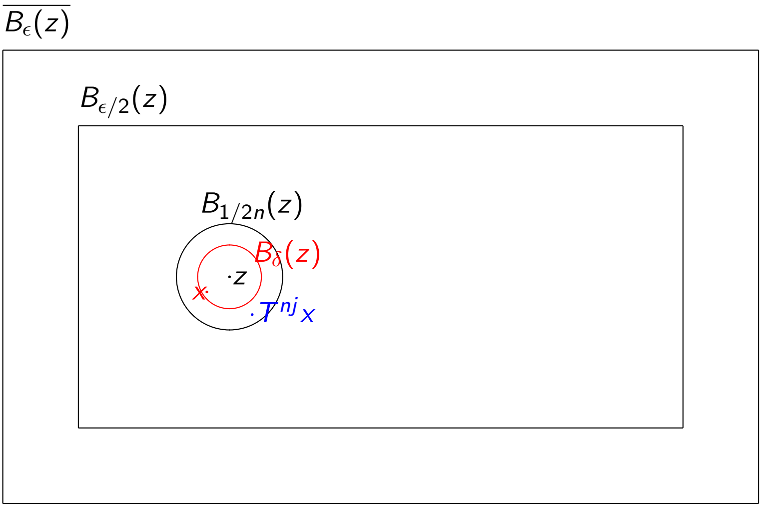
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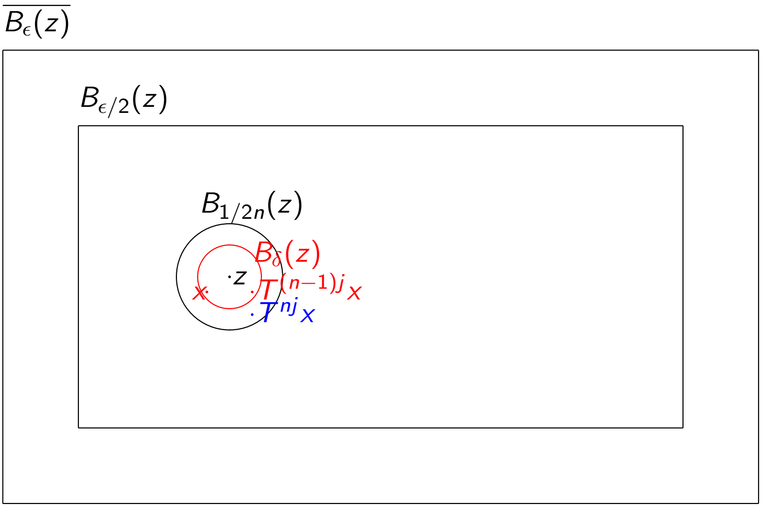
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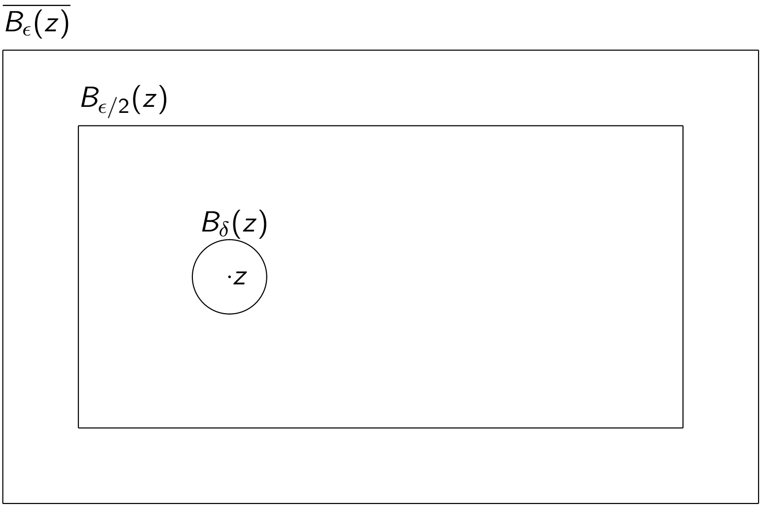
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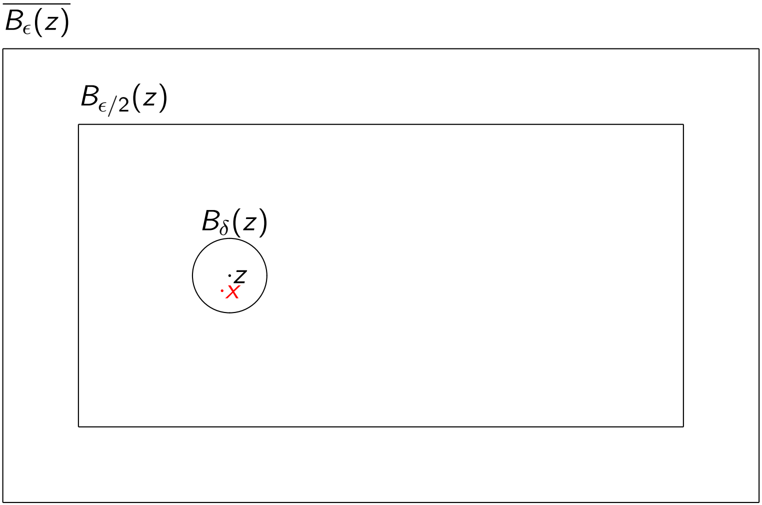
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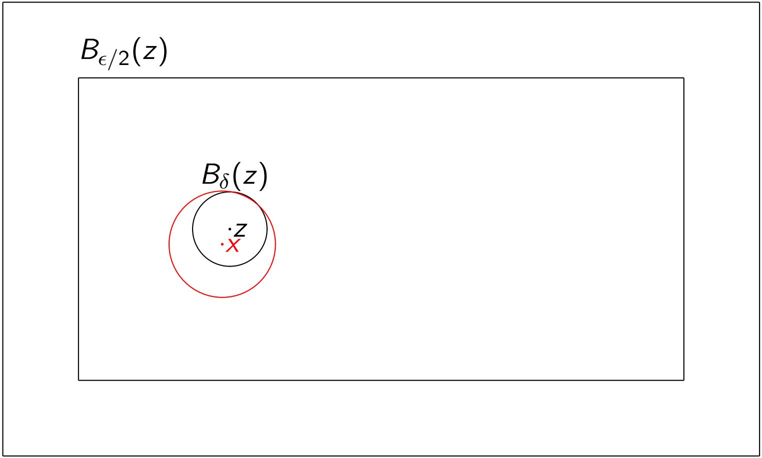

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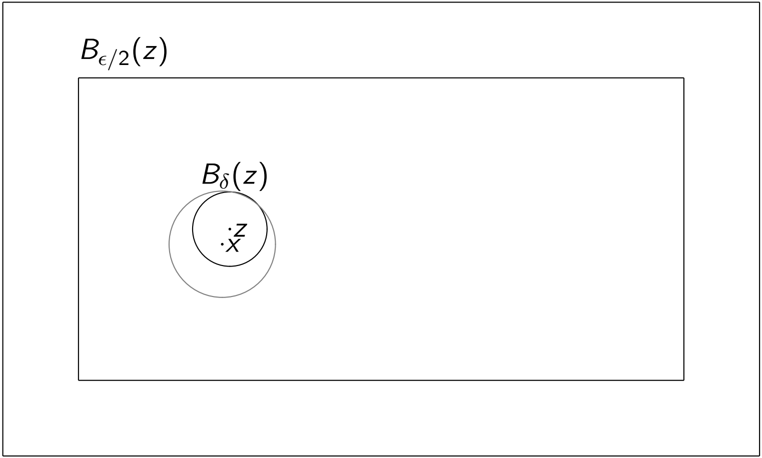

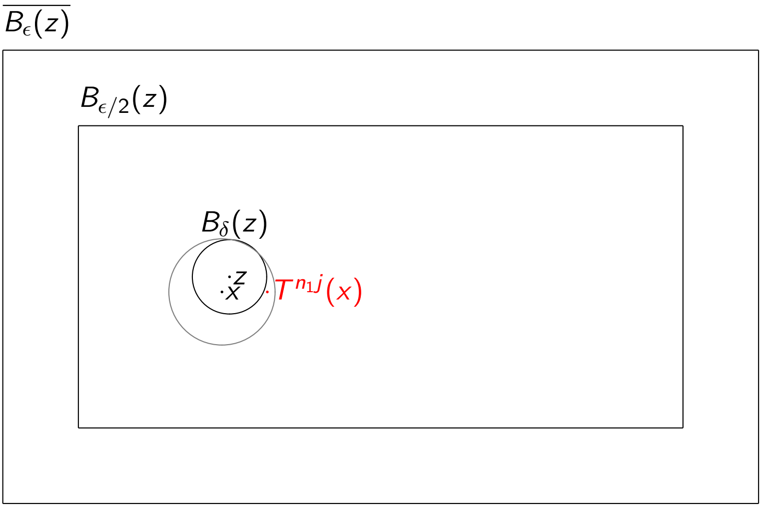
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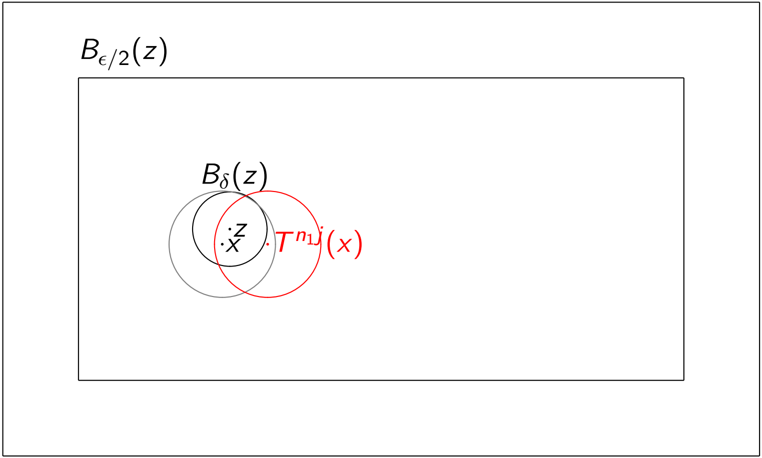
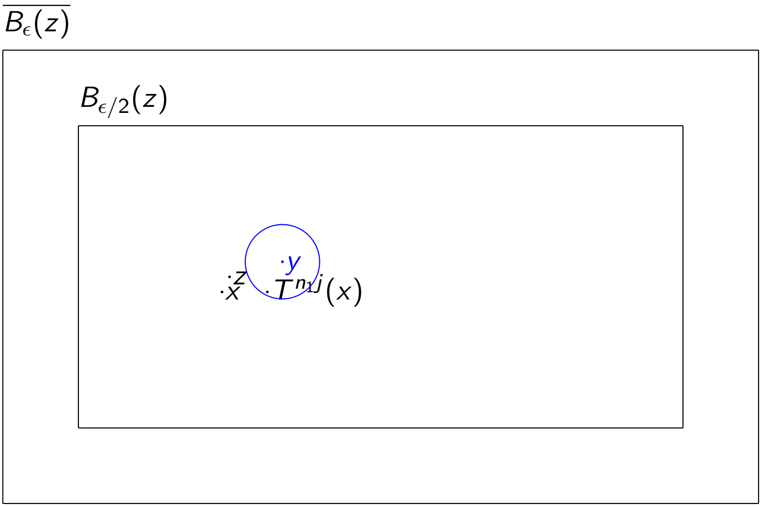


Where to now?

Notice that since $T^{n_1 j}(x) \notin B_\delta(z)$ we can't immediately continue.

Remember we're now looking inside $Y(\delta)$, so we know that $T^{n_1 j}(x)$ is within δ of some point in a j-cycle (say y), and we can thus continue, getting an n_2 so that

1. $d(T^{n_1 j}(x), T^{n_2 j}(x)) < 1/n$ and
2. $T^{n_2 j}(x)$ is more than δ from y.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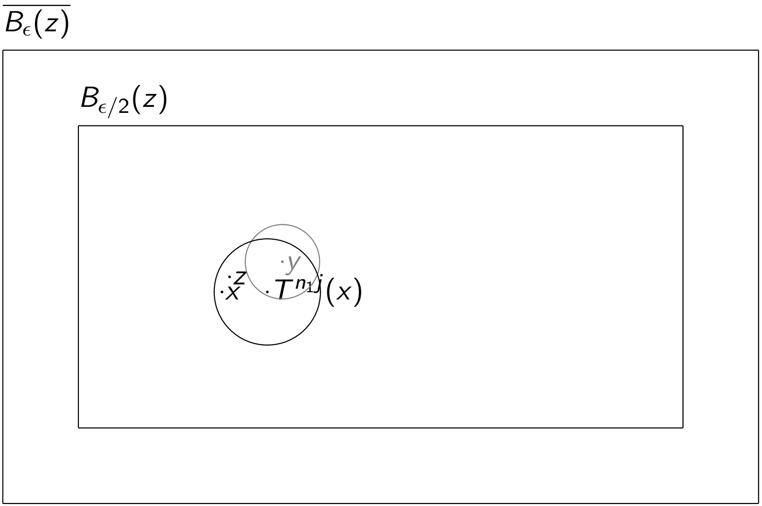
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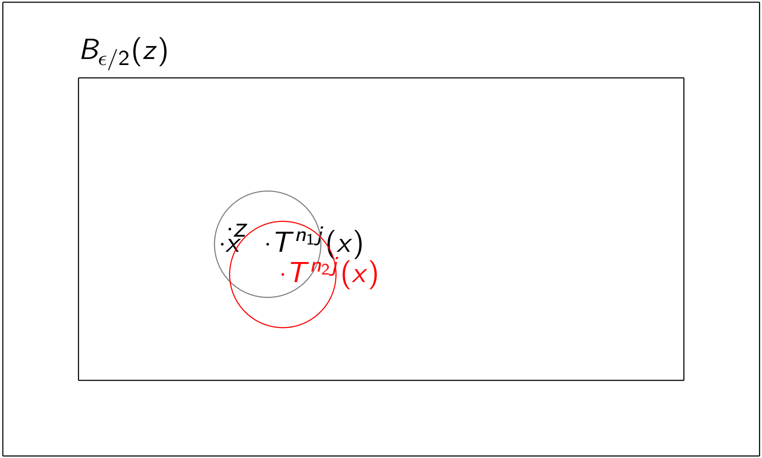

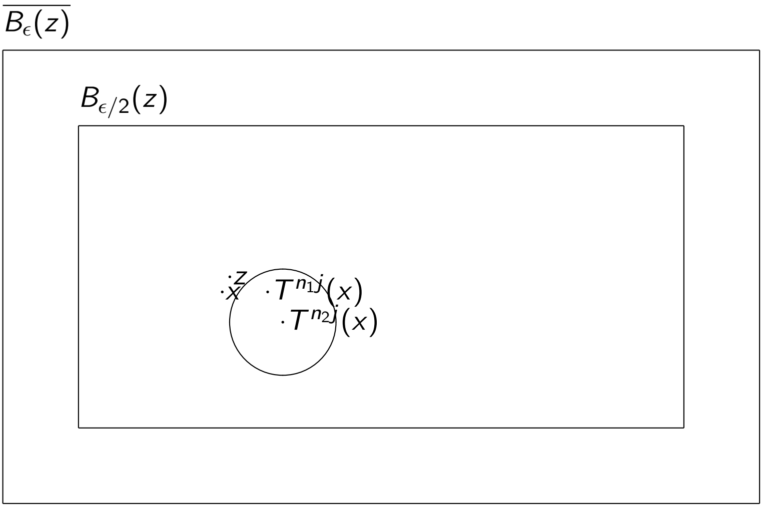
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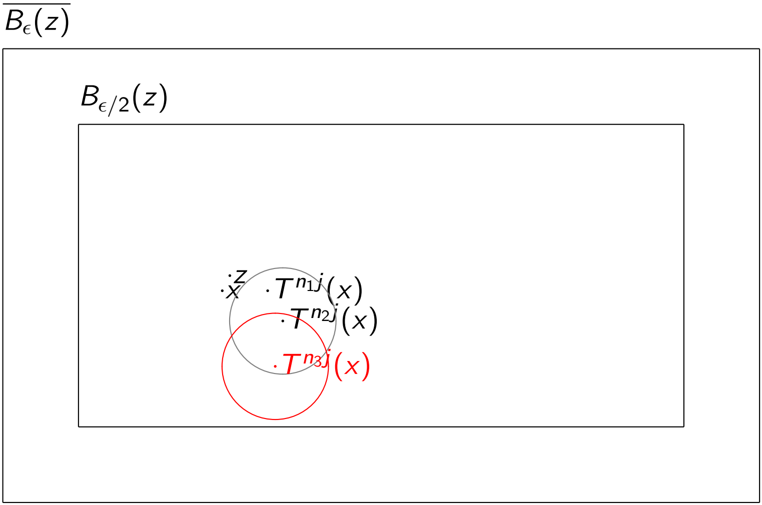
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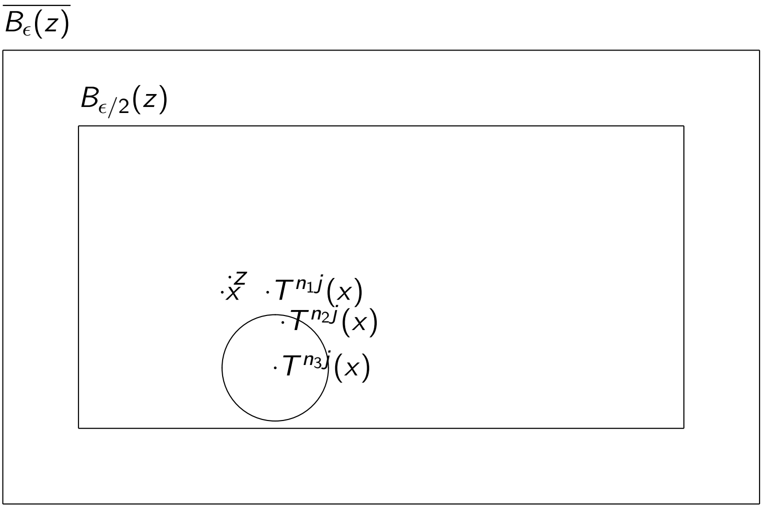
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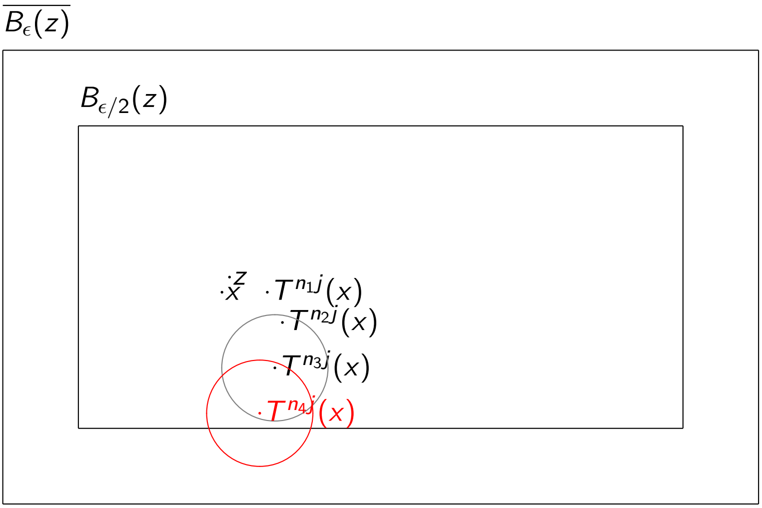
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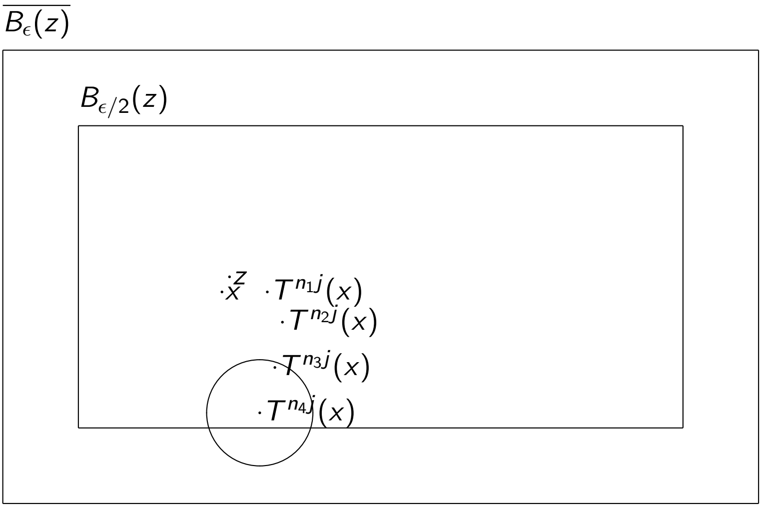
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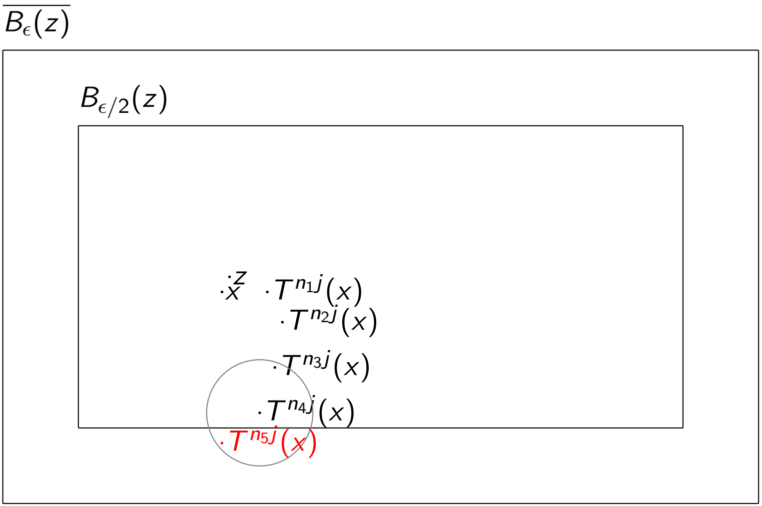
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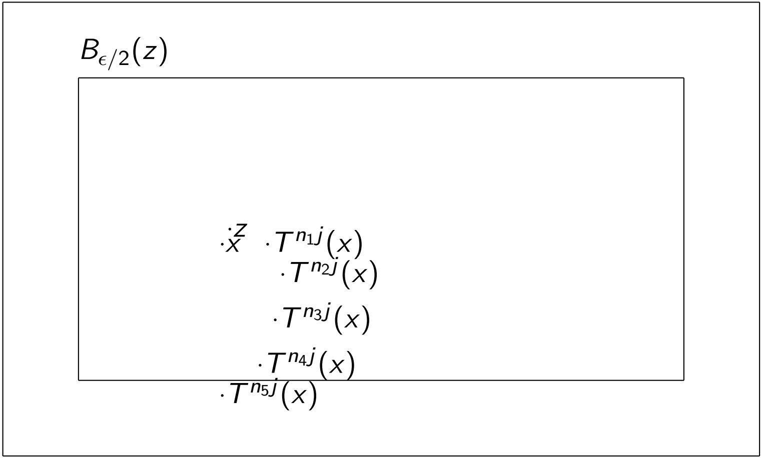

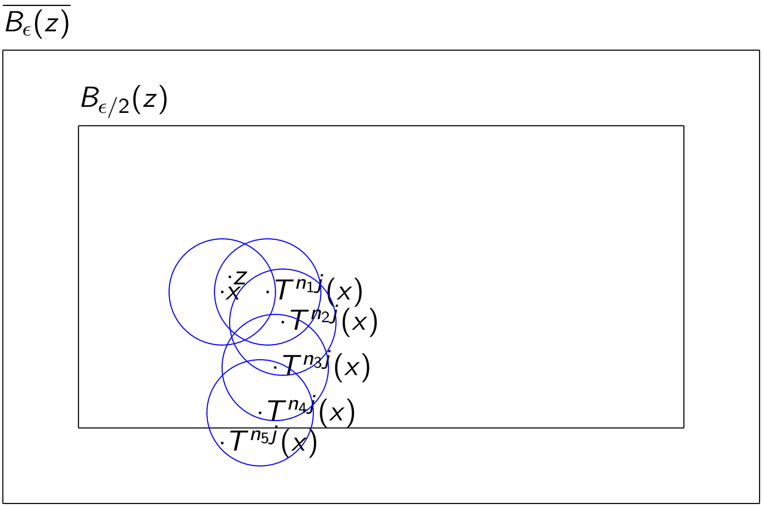
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3. For every $n \in \mathbb{N}$ there is a $1/n$-chain from z to q.

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Thus there is a connected component in Y (and in X) as desired.

Results

The previous method helps

Theorem

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1. $\zeta = \mathfrak{c}$,
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1. $\zeta = \mathfrak{c}$,
2. $1 \leq \zeta < \mathfrak{c}$, $\{n : \sigma_n = \mathfrak{c}\}$ *is infinite, and* $\sum\{\sigma_n : n \text{ is a stray period}\} \leq \zeta$, *or*
3. $\zeta = 0$, $\sigma(T)$ *is finitely based, and* $\sigma(T)$ *has no stray periods.*