Geometry of Cubics

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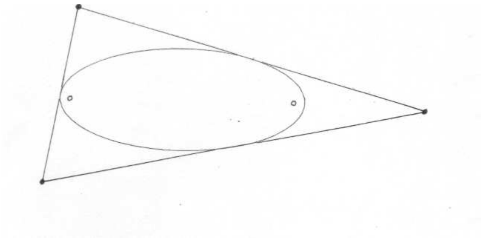
Marden’s theorem

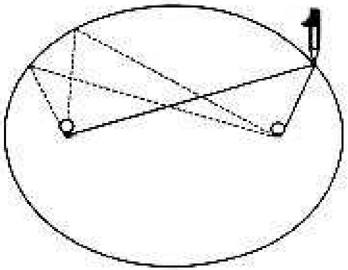
Roots of a general cubic (monic) polynomial $p(z)$ and the roots of its derivative $p'(z)$

Marden's theorem



The roots of $p'(z)$ are foci of "midpoint" ellipse corresponding to roots of $p(z)$.

Ellipses



$$\{x \in \mathbb{C} : |x - u| + |x - v| = L\}$$

Foci: u and v.

Linear Maps

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For some $a, b \in \mathbb{C}$,

$$z \mapsto az + b\bar{z}.$$

$$\left[= \frac{1}{2}[(\alpha + \delta + i(\gamma - \beta)](x + iy) + \frac{1}{2}[(\alpha - \delta + i(\gamma + \beta)](x - iy)\right]$$

Ellipses and Linear Maps

Every one-to-one linear map takes the unit circle to an ellipse

$$C := \{e^{i\theta} : 0 \leq \theta < 2\pi\}, E := \{ae^{i\theta} + be^{-i\theta} : 0 \leq \theta < 2\pi\}$$

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For $x \in E$*, let* $z := \sqrt{a}e^{i\theta/2}$ *and* $w := \sqrt{b}e^{-i\theta/2}$*.Then*

$$\begin{aligned} &|x - 2\sqrt{ab}| + |x + 2\sqrt{ab}| = |ae^{i\theta} + be^{-i\theta} - 2\sqrt{ab}| \\ &+ |ae^{i\theta} + be^{-i\theta} + 2\sqrt{ab}| = |z - w|^2 + |z + w|^2 \\ &= 2|z|^2 + 2|w|^2 = 2|a| + 2|b| \end{aligned}$$

and thus *E is an ellipse with foci $\pm 2\sqrt{ab}$.*

Given cubic $p(z)$ with roots $r + s + t = 0$

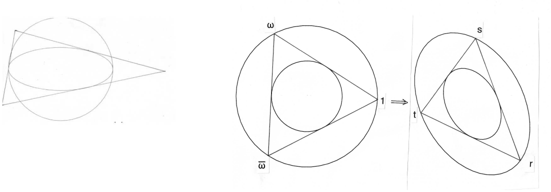

Figure: Projection of saturn-like figure implies there is a linear map

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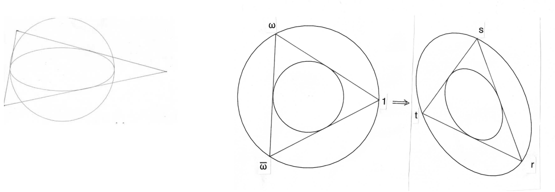

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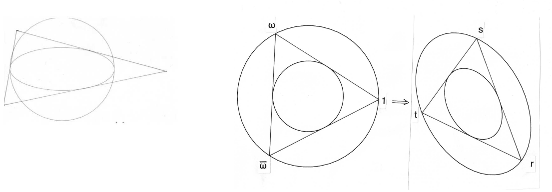

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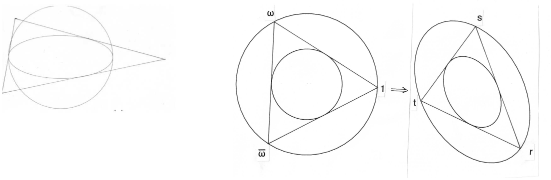

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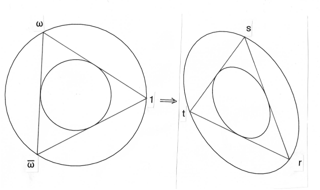
If $z=r,s$ or t, then, since $\omega=e^{2\pi i/3}$, $z^3-3abz=a^3+b^3$ and

$$p(z)=z^3-3abz-(a^3+b^3).$$

Proof of Marden's Theorem



Circles proportional by 2. Linear maps preserve midpoints.

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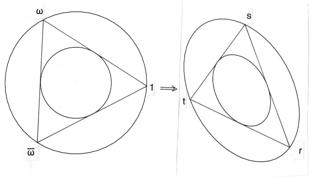

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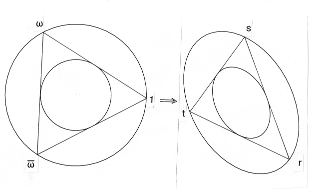

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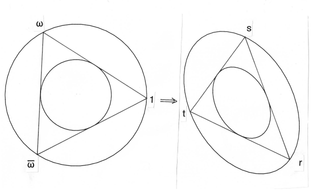

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Inner ellipse has foci $\pm\sqrt{ab}$ (since outer one has foci $\pm 2\sqrt{ab}$).
$p(z) = z^3 - 3abz - (a^3 + b^3)$ so $p'(z)$ has roots $\pm\sqrt{ab}$. QED

Cardano's formula

Given $p(z) := z^3 - 3Az - 2B$, we seek a, b so $ab = A$ and $a^3 + b^3 = 2B$.
Then $p(z) = z^3 - 3abz - (a^3 + b^3)$ which has roots $a + b, a\omega + b\overline{\omega}, a\overline{\omega} + b\omega$.

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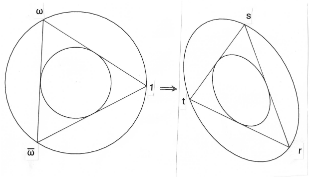
Then
$$(z - a^3)(z - b^3) = z^2 - 2Bz + A^3$$
and so
$$a^3, b^3 = B \pm \sqrt{B^2 - A^3}$$
$$a, b = \sqrt[3]{B \pm \sqrt{B^2 - A^3}}$$
chosen so $ab = A$ works.

Summary



- There is a linear map $z \mapsto az + b\overline{z}$ taking $1, \omega, \overline{\omega}$ to r, s, t.

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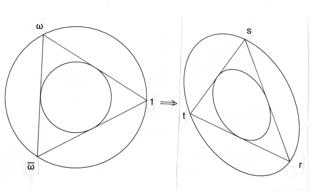

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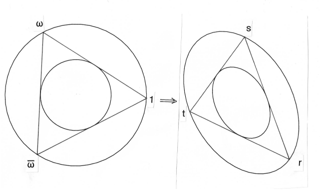

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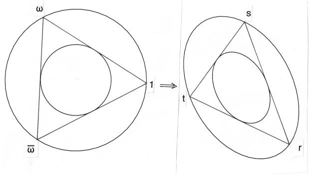

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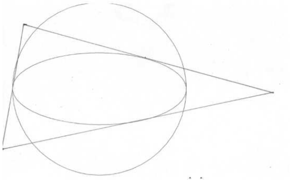
- $p(z) = z^3 - 3abz - (a^3 + b^3)$
- Foci of outer ellipse are $\pm 2\sqrt{ab}$ so roots of $p'(z)$, which are $\pm\sqrt{ab}$, are foci of inner ellipse.
- Finding a, b so that $p(z) = z^3 - 3abz - (a^3 + b^3)$ shows roots of $p(z)$ are $a + b, a\omega + b\overline{\omega}, a\overline{\omega} + b\omega$.

Real Case

Consider saturn-like figure



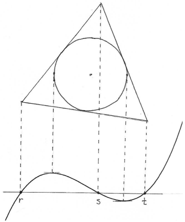


Figure: roots of p' correspond to endpoints of projection of inscribed sphere.