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Minimum-size convex decompositions in d dimensions

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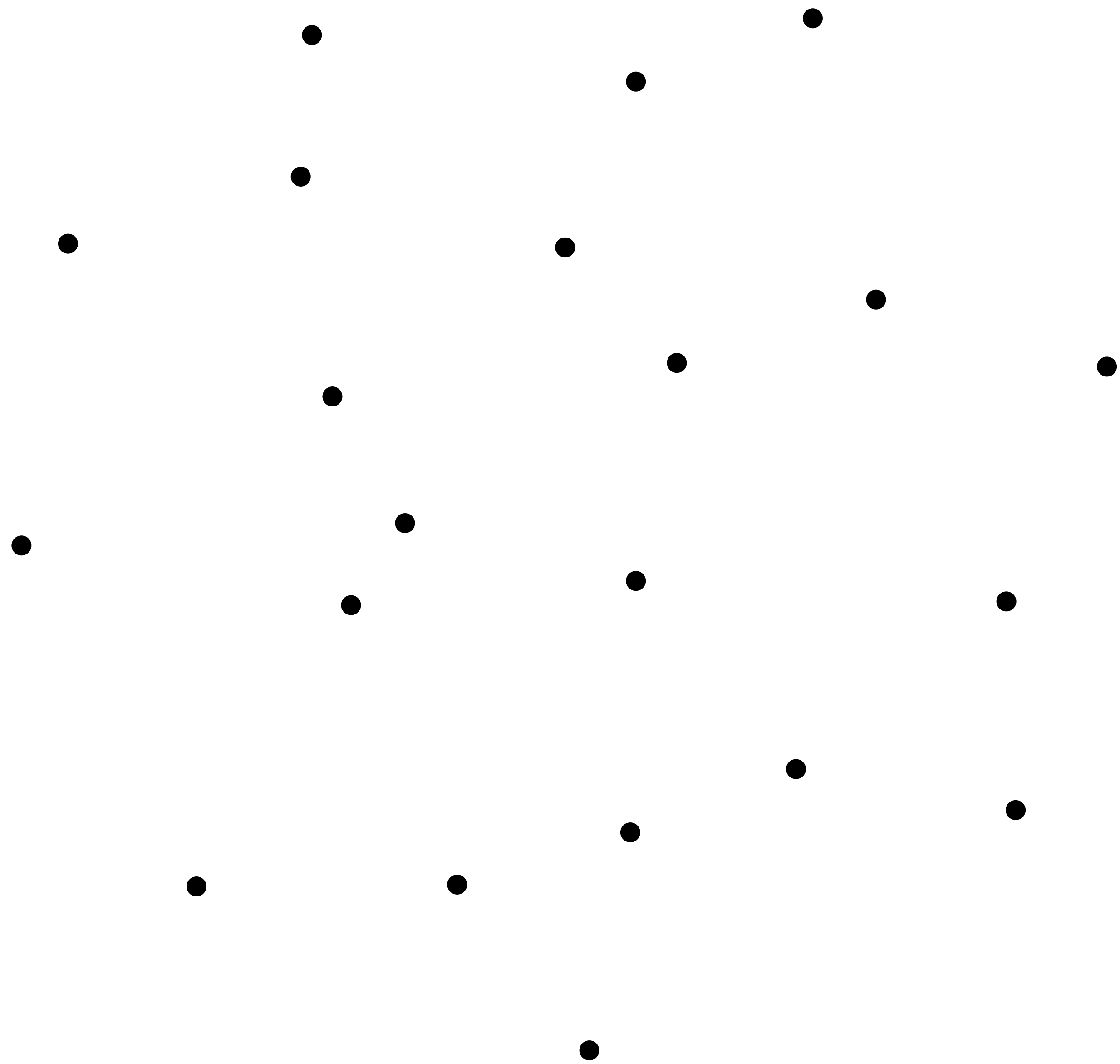
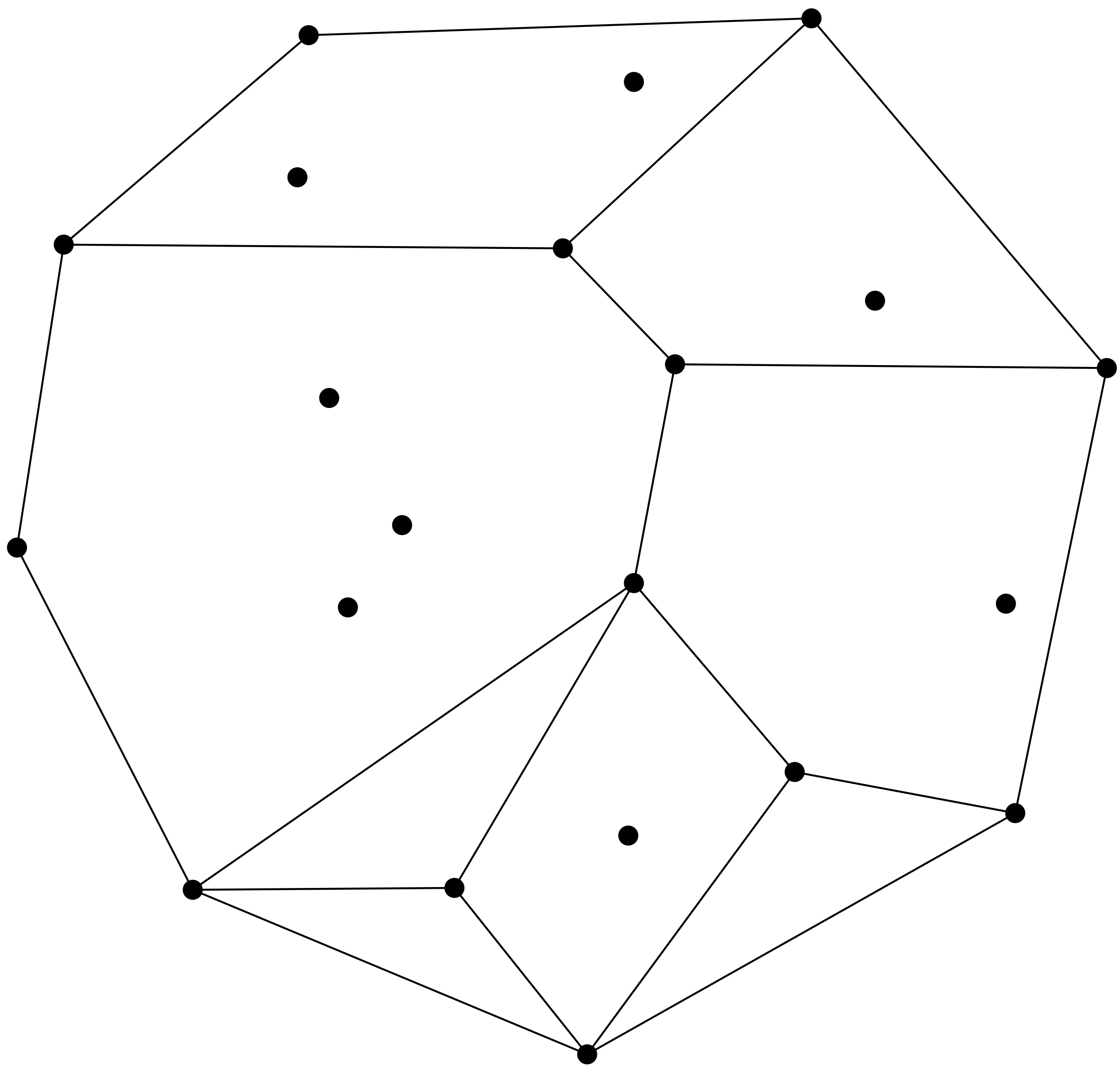
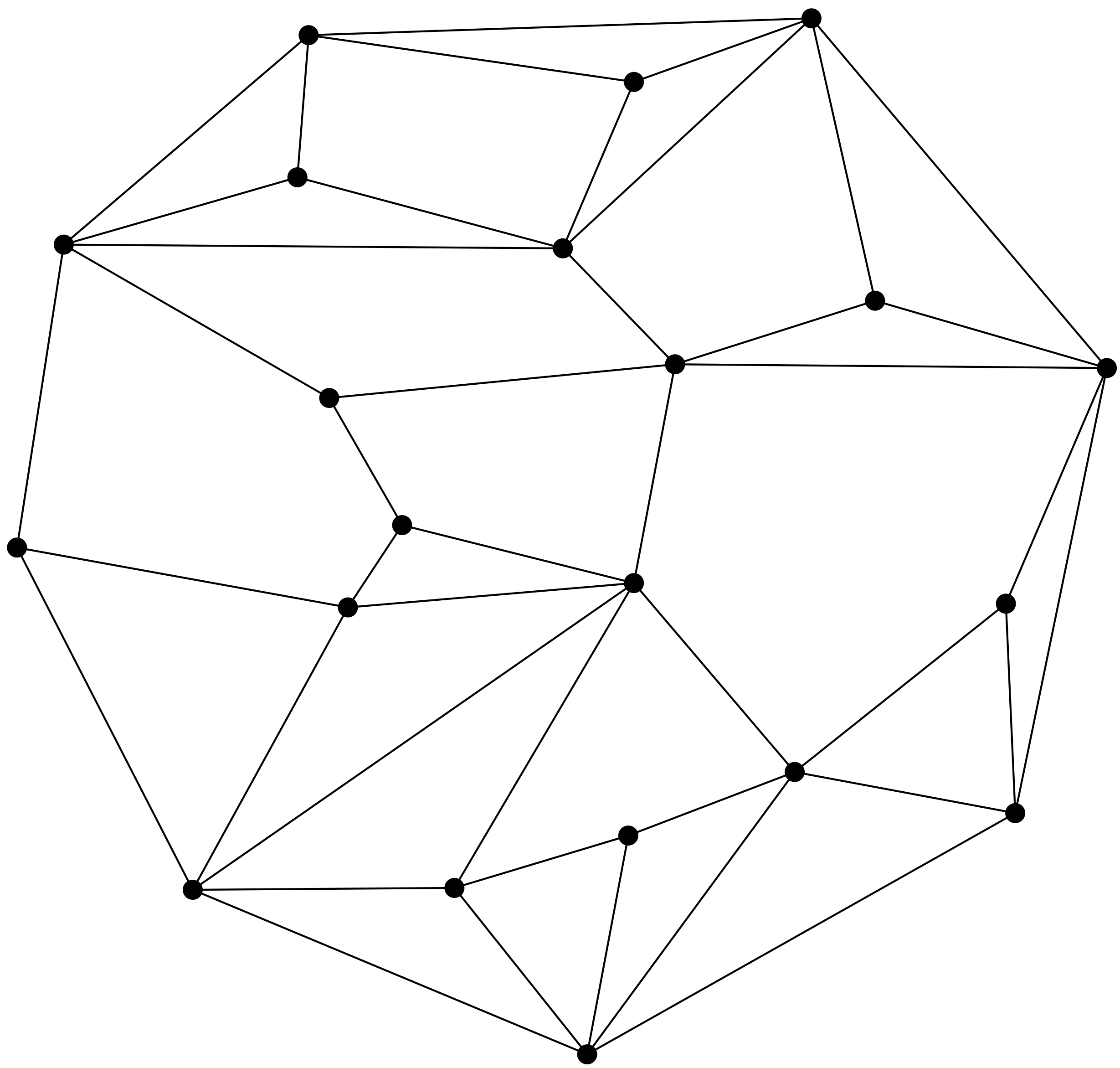


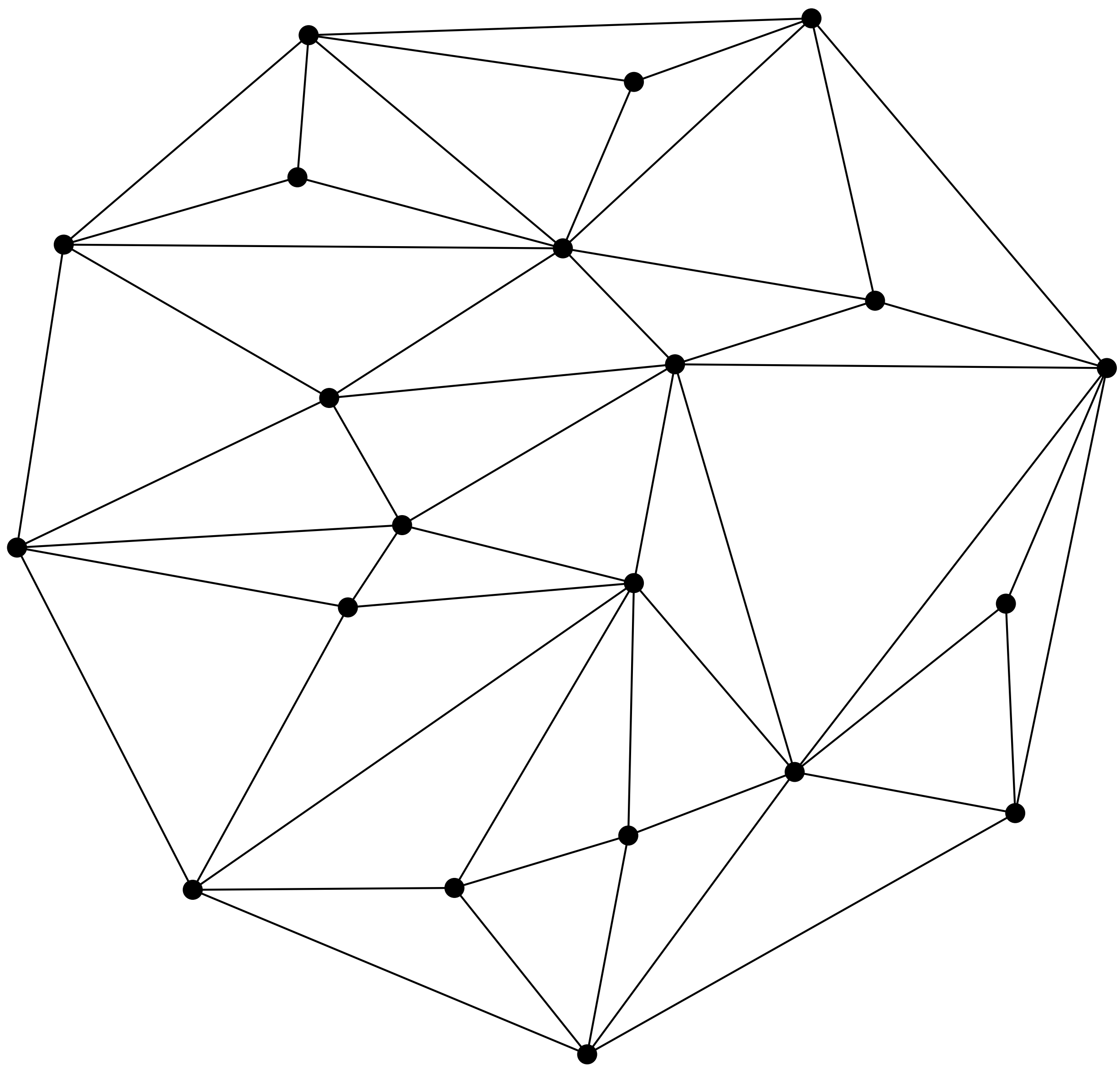
Consider a set V of n points in general position (i.e., every subset of $d+1$ points of V is affinely independent) in the d-dimensional Euclidean space.



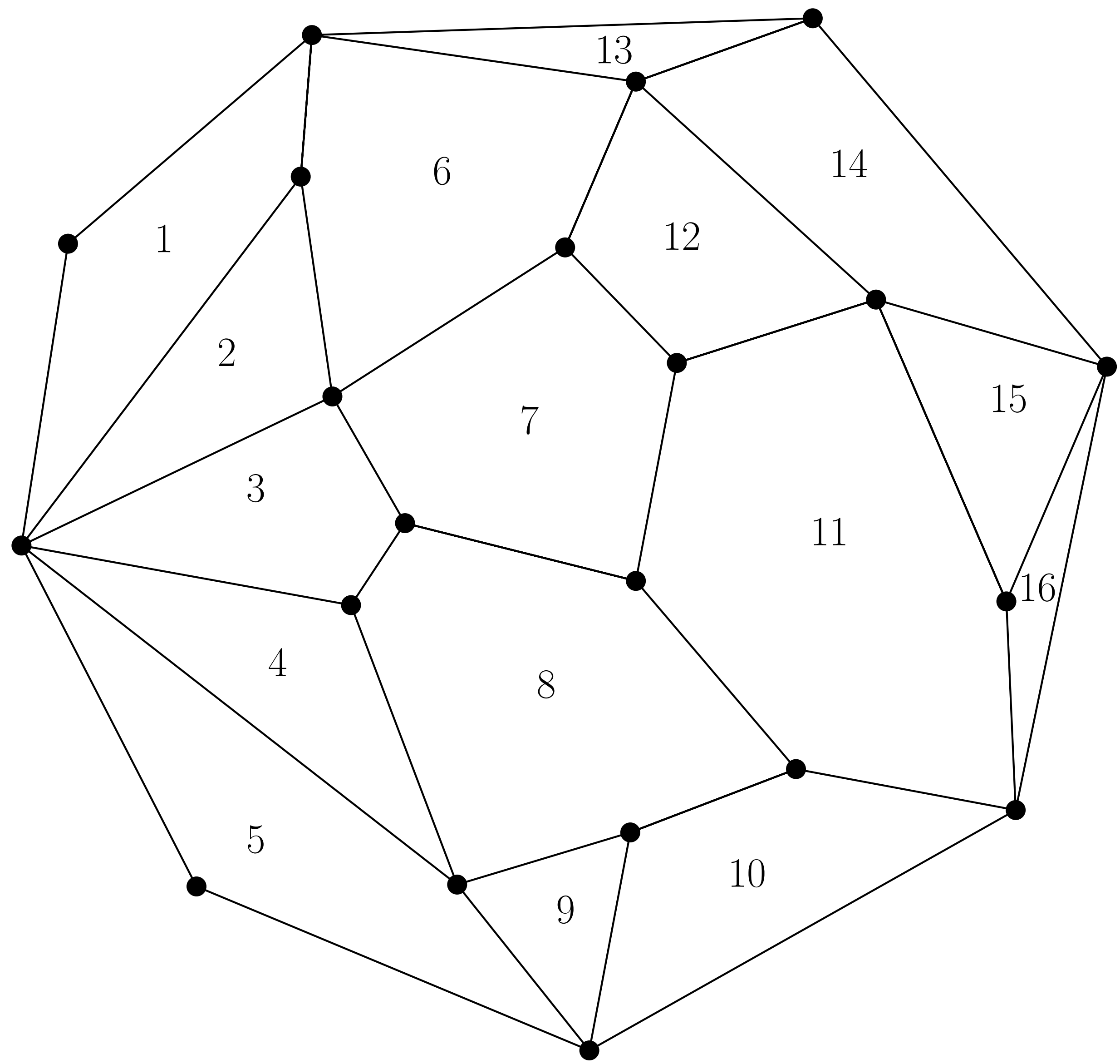
A *subdivision* of V is a set of convex polytopes $\{P_1, \ldots, P_k\}$ with vertices in V such that $\bigcup_i P_i = \mathrm{conv}(V)$ and $P_i \cap P_j$ is a (possibly empty) face of both P_i and P_j.



A *decomposition* of V is a subdivision $\{P_1, \ldots, P_k\}$ such that every point of V is a vertex for some P_i.



Note that a triangulation (i.e., a simplicial decomposition) of V is a decomposition.



The size of a decomposition is the number of polytopes in it. We want to find configurations of points for which the minimum size of a decomposition is as large as possible.

Let $G^d(V)$ be the minimum possible size of a decomposition of V.

Let $g^d(n)$ be the maximum value of $G^d(V)$ among the sets V of n points in general position in $\mathbb{R}^d$.

Let $g^d_{d+1}(n)$ be the maximum value of $G^d(V)$ among the sets V of n points in general position in $\mathbb{R}^d$ having exactly $d+1$ extreme points.

For $d = 2$ it was conjectured that $g(n) \leq n + C$ for some constant C. This conjecture is *false*.

Results from this talk:

$d = 2$ $\quad g_3(n) > n - 4 + \sqrt{2(n-3)}$ for $n \geq 3$.

$\quad g(n) > (35/32)n - 3/2$ for $n \geq 4$.

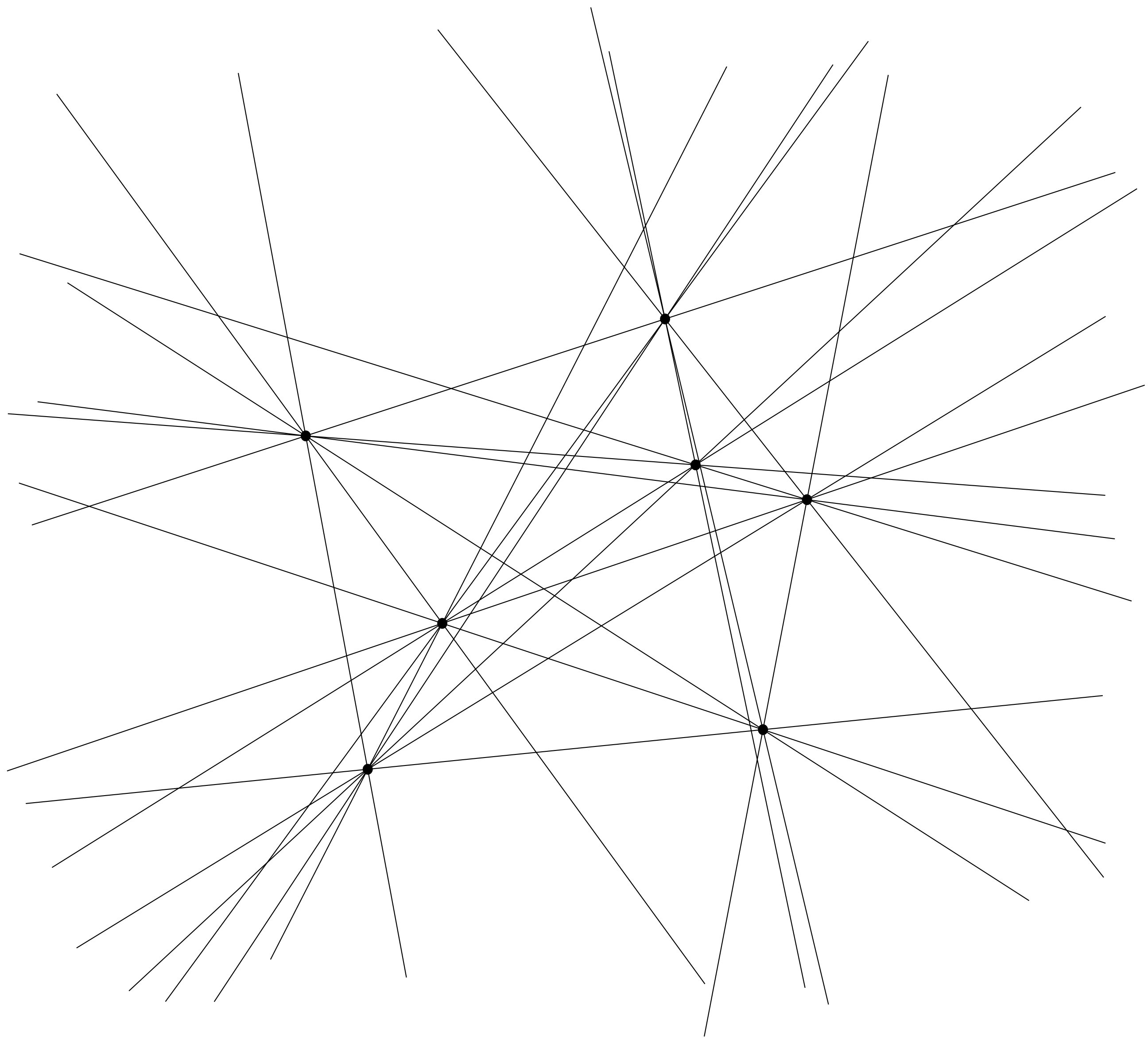
$d > 2$ $\quad g_{d+1} > n + \sqrt[d]{(d!)n} + C$

Previous work on upper bounds:

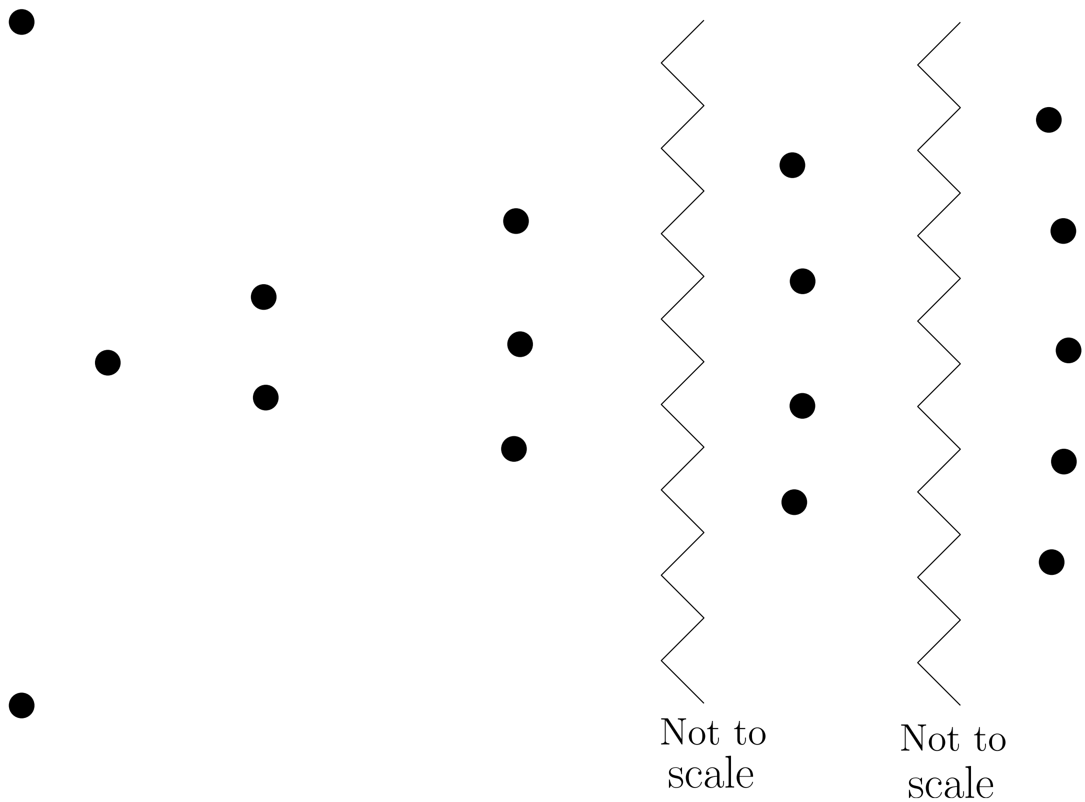
Urrutia (2004), $g(n) < (3/2)n + C$.

Hosono (2009), $g(n) < (7/5)n + C$.

Sakai and Urrutia (2009), $g(n) < (4/3)n + C$.

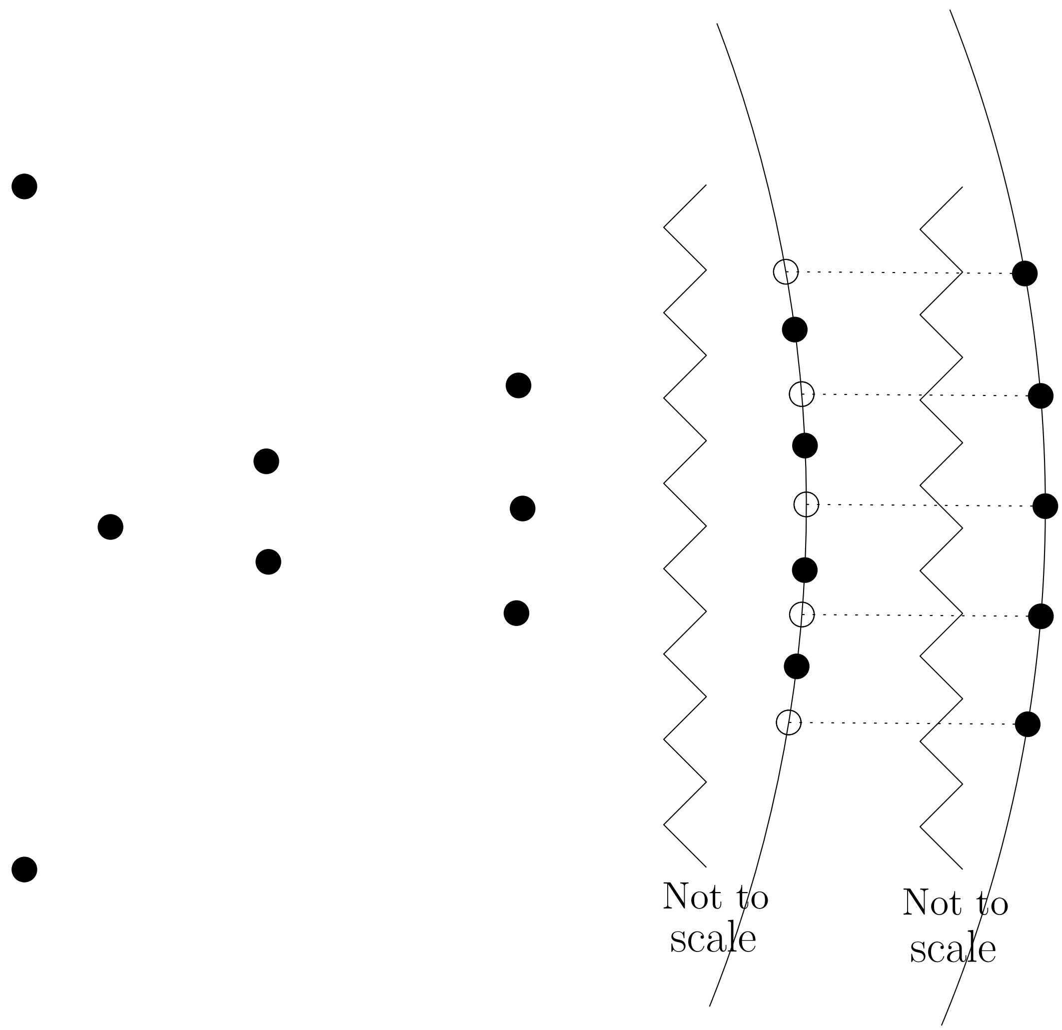


Given a vector $\vec{a}$ the cell at infinity in the hyperplane arrangement determined by V in the direction $\vec{a}$ is denoted by $\mathrm{cell}_\infty(\vec{a}, V)$.

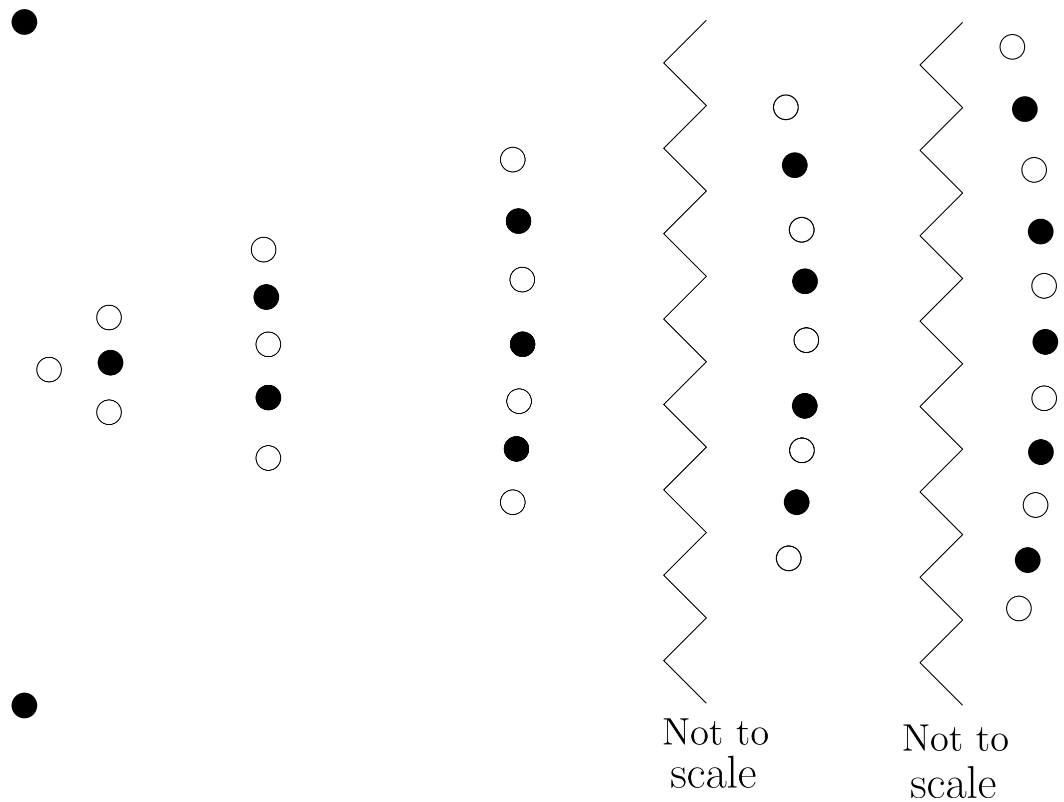


THE CONSTRUCTION ($d = 2$, THREE EXTREME POINTS)
The $k+1$ layer lies in the cell at infinity in the positive horizontal direction determined by the points in the previous layers.

Given a subdivision of the set with $k + 1$ layers, we obtain a subdivision of the set with k layers by contracting the $k + 1$ layer to infinity. It is easy to show by induction that a subdivision of minimum size for the set with $k + 1$ layers has $k + 2$ more polygons than a minimum-size subdivision for the set with k layers.



THE CONSTRUCTION ($d = 2$, THREE EXTREME POINTS)
Each layer lies on a smooth convex curve. The next layer is selected from the same curve and then shifted away until it lies in the cell at infinity determined by the previous layers.



THE CONSTRUCTION ($d > 2$, $d + 1$ EXTREME POINTS)
Each layer lies on a smooth convex $(d-1)$-surface. The next layer is selected by induction from the same surface and then shifted away until it lies in the cell at infinity.

Thank you.