# The Log-Convex Density Conjecture and tangential surface area in $\mathbb{R}^{n}-\{0\}$ 

 (or vertical surface area in warped products). arXiv:1107.4402
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## The isoperimetric problem in manifolds with density

Problem: In a Riemannian manifold $M$ equipped with a positive function $\Psi_{S}$ weighting surface area and a positive function $\Psi_{V}$ weighting volume, which region has the least weighted surface area among all regions of a fixed weighted volume?

## Definition

A region is isoperimetric if it has the least weighted surface area among all regions of the same weighted volume. A surface is isoperimetric if it bounds an isoperimetric region.

## Examples

We are most interested in radial densities.

## Example

Some known results:

- In $\mathbb{R}^{n}$ with simple density $\Psi_{S}=\Psi_{V}=e^{r^{2}}$, spheres about the origin are isoperimetric (Rosales, et al. [4])
- In $\mathbb{R}^{n}-\{0\}$ with simple density $\Psi_{S}=\Psi_{V}=r^{-p}, p>n$, spheres about the origin are isoperimetric, bounding volume at infinity (Díaz, et al. [1])
- In $\mathbb{R}^{n}-\{0\}$ with any simple increasing radial density $\Psi_{S}=\Psi_{V}=e^{\phi(r)}$ with $\phi^{\prime \prime}(r)>\epsilon>0$, large spheres about the origin are isoperimetric. (Kolesnikov and Zhdanov [2]).


## The log-convex density conjecture

Conjecture. In $\mathbb{R}^{n}, n \geq 2$ with radial log-convex simple density, balls about the origin are isoperimetric for every volume. (Rosales, et al. [4, Conj. 3.12])

## Fact

Log-convexity on $\mathbb{R}^{n}-\{0\}$ is equivalent to the stability of spheres about the origin (stability means that they are local minimizers up to order 2). However, there are simple examples of densities on $\mathbb{R}^{n}$ that are log-convex on $\mathbb{R}^{n}-\{0\}$ but neither log-convex nor smooth at the origin and where spheres (or some spheres) are not isoperimetric (see Morgan [3]).

## Generalizing the log-convex density conjecture

- Log convexity at the origin is a strong restriction on simple radial densities that are log-convex on $\mathbb{R}^{n}-\{0\}$.
- No decreasing simple radial density log-convex on $\mathbb{R}^{n}-\{0\}$ is log-convex (or even defined) at the origin
- There are, however decreasing densities where spheres about the origin are isoperimetric (see example 2 from the prior slide).
- Problem: Give sufficient conditions on a simple density on $\mathbb{R}^{n}-\{0\}$ for spheres to be isoperimetric (log-convexity is necessary, what else?).


## Tangential surface area

We give a partial answer to this problem by studying when spheres about the origin in $\mathbb{R}^{n}-\{0\}$ minimize "tangential" surface area, that is, the component of surface area tangential to spheres about the origin. (In the more general setting of warped products we call this vertical surface area because it is the component of surface area in the second, "vertical" component of the product).

- For a sphere about the origin $S,|S|_{\text {tan }}=|S|$.
- For any rectifiable hypersurface $H,|H|_{\tan } \leq|H|$.
- If a sphere about the origin minimizes tangential surface area, it is isoperimetric.


## Our main result

Our main result is a sufficient condition for a specific sphere about the origin in $\mathbb{R}^{n}-\{0\}$ with arbitrary continuous radial densities to minimize vertical surface area among surfaces bounding the same volume.

## Theorem

Let $\Phi$ be the function sending the weighted volume of a sphere about the origin to its weighted surface area. If there exists a volume $V_{0}$ and a function $\tilde{\Phi}$ such that:

- $\tilde{\Phi}\left(V_{0}\right)=\Phi\left(V_{0}\right)$ and $\tilde{\Phi}(0)=0$
- $\tilde{\Phi} \leq \Phi$
- $\tilde{\Phi}$ is convex
then the sphere about the origin bounding weighted volume $V_{0}$ minimizes tangential surface area among hypersurfaces bounding the same weighted volume and is, in particular, isoperimetric.


$$
\Psi_{V}=e^{r}, \Psi_{S}=e^{r^{8}}
$$

## Corollaries

- Gives a generalization and new proof of the work of Kolesnikov and Zhdanov on large spheres: our result implies that large spheres are isoperimetric for eventually strictly log-convex radial simple densities on $\mathbb{R}^{n}$ (including, for example, the density $e^{p(r)}$ where $p$ is a polynomial of degree $\geq 2$ with positive leading coefficient).
- Generalizes work of Díaz et al. to give new examples of decreasing simple radial densities where spheres about the origin are isoperimetric (bounding volume at infinity): $r^{p} e^{\phi(r)}$ with $\phi^{\prime \prime} \geq 0$ and either $\phi^{\prime}(r) \leq 0$ and $p<-n$ or $\phi^{\prime}(r) \leq-\epsilon<0$ and $p=-n$.
- Our methods give similar results in arbitrary warped products of Euclidean intervals and compact manifolds with continuous product densities (not necessarily simple density).
- Change of coordinates to move into the simpler case of constant volume density
- A simple convexity argument (the same as that of Díaz et al. for density $r^{p}, p<-n$ ).
- Some technical work to deal with different possible volume conditions (finite/infinite volume at the origin/infinity) and hypersurfaces that don't separate the origin from infinity.


## Conclusion

- Paper available upon request
- Sean Howe, "The Log-Convex Density Conjecture and vertical surface area in warped products."
- arXiv:1107.4402
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- Questions? (If time.)
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